Solution of Exercise 1

(a) Take $x_1 = v, x_2 = \dot{v}$, then $\dot{x} = f(x,u), y = g(x,u)$ with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad f(x,u) = \begin{pmatrix} x_2 \\ -x_1 + x_2^3 + u^3 \end{pmatrix}, \quad g(x,u) = x_1^2 + u^2.$$

(b) By substitution it follows that $u(t) = -2\cos t$, $y(t) = 4$.

(c) Using some alternative notation, $\dot{\Delta x} = A(t)\Delta x + B(t)\Delta u, \Delta y = C(t)\Delta x + D(t)\Delta u$ with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 12\cos^2 t \end{pmatrix}, \; B(t) = \begin{pmatrix} 0 \\ 12\cos^2 t \end{pmatrix}, \; C(t) = \begin{pmatrix} 4\sin t & 0 \end{pmatrix}, \; D(t) = -4\cos t.$$

Solution of Exercise 2

(a) The transfer function of the interconnection equals $\dfrac{(s+1)(s+4)}{s^3+4s^2+10s+4+\alpha}$.

(b) Using the Routh table it follows easily that for $\alpha \in (-4, 36)$ the transfer function is stable. There may be pole-zero cancellation for $s = -1$ or $s = -4$, so that only stable poles may be cancelled. Hence, the conclusion remains that for $\alpha \in (-4, 36)$ the transfer function is stable.

(c) For $\alpha = 3$, the transfer function reduces to $\dfrac{(s+4)}{s^2+3s+7}$. A realisation is given by $A = \begin{pmatrix} 0 & 1 \\ -7 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 1 \end{pmatrix}$, $D = 0$. It follows easily that the realisation is controllable and observable, so that it has a minimal dimension.

Solution of Exercise 3

(a) The characteristic equation equals $\det(\lambda I - A) = \lambda^3 + \lambda^2 + \lambda\beta^2 + \beta^2 = (\lambda+1)(\lambda^2+\beta^2)$. Hence, the eigenvalues are $\lambda_1 = -1, \lambda_{2,3} = \pm i\beta$. If $\beta \neq 0$, then all eigenvalues are distinct in the closed left half plan, indicating that the system is stable for $\beta \neq 0$. If $\beta = 0$, then there are two eigenvalues at zero on the imaginary axis with an eigenspace that can shown to be one dimensional, implying that the system is unstable for $\beta = 0$.

(b) The controllability matrix is $R = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\beta^2 \\ 0 & -1 & 0 \\ 2 & 0 & -2\beta^2 \end{pmatrix}$.
Clearly, $\det R = 0$ for all β. So, the system is not controllable for any β.
The controllable subspace is given by $\operatorname{im} R = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(c) Extend the basis of the controllable subspace, to obtain the transformation matrix $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$. Transformation to the associated basis yields

$\tilde{A} = T^{-1}AT = \left(\begin{array}{cc|c} 0 & \beta^2 & 0 \\ -1 & 0 & -\frac{1}{2} \\ \hline 0 & 0 & -1 \end{array}\right)$, $\tilde{B} = T^{-1}B = \left(\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right)$ and $\tilde{C} = CT = \left(\begin{array}{cc|c} 10 & 1 & 4 \end{array}\right)$. Note that $\left(\begin{pmatrix} 0 & \beta^2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ is a controllable pair.

Solution of Exercise 4

Note that the pair (A, B) has already the well-known controllable/non-controllable decomposition. Hence, the eigenvalues in the upper left part can be assigned by feedback and the eigenvalues in the lower right part are fixed. Also note that with B only the second row of matrix A can be modified.

(a) Take $F = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \end{pmatrix}$ and observe by looking at the matrices that $\det(\lambda I - (A+BF)) = \det \begin{pmatrix} \lambda & -1 \\ 1 - f_1 & \lambda - 3 - f_2 \end{pmatrix} \det \begin{pmatrix} \lambda + 1 & 4 \\ -1 & \lambda + 1 \end{pmatrix}$ $= \Big(\lambda^2 - (3 + f_2)\lambda + (1 - f_1)\Big)\Big((\lambda + 1)^2 + 4\Big)$. The eigenvalues characterised by $\Big((\lambda + 1)^2 + 4\Big)$ are fixed and are located at $-1 \pm 2i$ in the open left half plane. The eigenvalues characterised by $\Big(\lambda^2 - (3 + f_2)\lambda + (1 - f_1)\Big)$ can be assigned arbitrarily, especially in the open left half plane. Hence, the system is stabilizable.

(b) From the question it follows that $q(s)$ is a feasible characteristic polynomial of $A+BF$ for suitable F. Hence, it must contain $\Big((\lambda+1)^2+4\Big)$ as a factor. In fact, it follows by long division that $q(s) = \Big(\lambda^2+4\lambda+3\Big)\Big((\lambda+1)^2+4\Big)$. From part (a) it follows that then f_1 and f_2 can be chosen as $-(3+f_2) = 4$ and $(1-f_1) = 3$, or $f_1 = -2$ and $f_2 = -7$. The elements f_3 and f_4 can be chosen arbitrarily. Hence, $F = \begin{pmatrix} -2 & -7 & * & * \end{pmatrix}$, with the $*$'s denoting arbitrary values, answers the question.

(c) From part (a) it follows that the possible characteristic polynomials are $(\lambda^2 + a\lambda + b)((\lambda + 1)^2 + 4)$, with $a, b \in \mathbb{R}$ free.

Solution of Exercise 5

(a) According to the Hautus test, the pair (C, A) is observable if and only if $\operatorname{rank} \begin{pmatrix} A - sI \\ C \end{pmatrix} = n$ for all $s \in \mathbb{C}$, which means that the $(n+p)\times n$ matrix $\begin{pmatrix} A_{11} - sI & 0 \\ 0 & A_{22} - sI \\ C_1 & C_2 \end{pmatrix}$ has full column rank $n_1 + n_2$ for all $s \in \mathbb{C}$. Hence, it follows that $\operatorname{rank} \begin{pmatrix} A_{11} - sI \\ 0 \\ C_1 \end{pmatrix} = n_1$ and $\operatorname{rank} \begin{pmatrix} 0 \\ A_{22} - sI \\ C_2 \end{pmatrix} = n_2$ for all $s \in \mathbb{C}$. Since the zero matrices do not matter here, it follows that $\operatorname{rank} \begin{pmatrix} A_{11} - sI \\ C_1 \end{pmatrix} = n_1$ and $\operatorname{rank} \begin{pmatrix} A_{22} - sI \\ C_2 \end{pmatrix} = n_2$ for all $s \in \mathbb{C}$. In other words, the pairs (C_1, A_{11}) and (C_2, A_{22}) are observable.

(b) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Then $A_{11} = A_{22} = C_1 = C_2 = 1$. It follows directly that (C_1, A_{11}) and (C_2, A_{22}) are observable. However, it is seen easily that (C, A) is not observable.

(c) Let λ be an arbitrary eigenvalue of A_{11}, then A_{22} is invertible and it follows that $\begin{pmatrix} A_{11} - \lambda I & 0 \\ 0 & A_{22} - \lambda I \\ C_1 & 0 \end{pmatrix} = n_1 + n_2$. Using contradiction, it can even be shown that also the matrix $\begin{pmatrix} A_{11} - \lambda I & 0 \\ 0 & A_{22} - \lambda I \\ C_1 & C_2 \end{pmatrix}$ has rank $n_1 + n_2$, implying that $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ has rank n. Since the same holds for any eigenvalue of A_2, it follows by the Hautus test that the pair (C, A) is observable.

Solution of Exercise 6

(a) By repeated application of the Cayley Hamilton theorem, it follows that A^k can be expressed as a linear combination with scalar coefficients of $A^0, A^1, \ldots, A^{n-1}$, for all $k \geq n$. Hence, CA^kB can be expressed as a linear combination with scalar coefficients of $CB, CAB, \ldots, CA^{n-1}B$, for all $k \geq n$. The same then applies to $Ce^{At}B = \sum_{k \geq 0} \frac{CA^kBt^k}{k!}$. Hence, to compute the impulse response $Ce^{At}B$, only $CB, CAB, \ldots, cA^{n-1}B$ are really required, and the statement is true.

(b) If (A, B) is controllable, then rank $(A - sI \;\; B) = n$ for all $s \in \mathbb{C}$. Hence, certainly for $s = 0$ it then follows that rank $(A \;\; B) = n$, or im $(A \;\; B) = \mathbb{R}^n$. If rank A + rank $B < n$, then dim im A + dim im $B < n$, implying that dim im $(A \;\; B) \leq$ dim im A + dim im $B < n$, and consequently (A, B) can not be controllable. Hence, the statement is true.

(c) The statement is false. Take $n = p =$ and $A = C = 1$, then (C, A) and (C, A^{-1}) are both discrete time observable, and therefore also discrete time detectable.