Resit *Linear Algebra* (CSE1205), July 4, 2019, 13:30-16:30

- Calculators and formula sheets are **not** allowed.
- Credits: **2 points** for questions from Part I (19 questions) and **4 points** for questions from Part II (3 questions).
- The final score: Sum and divide by 5.

PART I: MULTIPLE CHOICE QUESTIONS

1. Let (x_1, x_2, x_3) be the unique solution of the system

$$\begin{aligned} x_2 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_1 \quad + \quad 5x_3 &= -2 \end{aligned}$$

Then x_1 is equal to:

A. -3 **B.** -2 **C.** -1 **D.** 0 **E.** 1 **F.** 2 **G.** 3 **H.** 4

Answer: G.

The system is consistent, with unique solution $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$.

2. The dimension of Nul(A), where $A = \begin{bmatrix} 0 & 1 & 2 & -2 & -1 & 3 & 0 \\ 1 & 3 & 1 & 1 & 2 & 0 & 0 \\ -1 & 3 & 4 & 2 & -2 & -1 & 0 \end{bmatrix}$, is given by:

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4 **F.** 5 **G.** 6 **H.** 7

Answer: E.

A has 3 pivot positions, whence the number of free variables is equal to $7 - 3$.

3. It is given that

$$A = \begin{bmatrix} \| & \| & \| & \| & \| \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \| & \| & \| & \| & \| \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following sets can be taken as a basis for Col A (for any matrix A satisfying the above condition)?

(I) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

(II) $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5\}$

A. None
B. Only (I)
C. Only (II)
D. (I) and (II)

Answer: C.

(I) is not always true.

A counterexample is given by:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(II) is always true.

The rank of A is 3, so Col(A) has dimension 3. Since the columns $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5$ are linearly independent (just observe that $[\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5]$ has rank 3), they must be a basis of Col(A) by the basis theorem.

4. The solution set of the system $A\mathbf{x} = 0$ has a basis that consists of four vectors and A is a 7×9-matrix. What is the rank of A?

A. 1 **B.** 2 **C.** 3
D. 4 **E.** 5 **F.** 6
G. 7 **H.** There is no sufficient information to determine the rank

Answer: E.

A has 9 columns, so its rank is equal to 9 minus the dimension of Nul(A) by the Rank Theorem, i.e. $9 - 4$.

5. Suppose that X, Y, Z are 3×3 matrices such that $XYZ = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. Which of the matrices must be invertible?

A. None **B.** Only X **C.** Only Y **D.** Only Z
E. Only X,Y **F.** Only X,Z **G.** Only Y,Z **H.** X,Y,Z

Answer: H.

$|X||Y||Z| = |XYZ| = 1 \cdot 4 \cdot 6 \neq 0 \implies |X|, |Y|, |Z| \neq 0 \implies X, Y, Z$ are invertible.

6. Find the determinant det(A) of $A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$:

A. -8 **B.** -6 **C.** -4 **D.** -2 **E.** 0 **F.** 2 **G.** 4 **H.** 6

Answer: D.

Expand along the last column or row-reduce to an echelon form: $\det(A) = -2$.

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For the following two questions, let $A = LU$ be the LU-decomposition of the matrix

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$$

7. The first column of L is equal to

A. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ **B.** $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ **D.** $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ **E.** $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ **F.** $\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$ **G.** $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ **H.** $\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$

Answer: A. $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$

8. The last column of U is equal to

A. $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ **B.** $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **D.** $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ **E.** $\begin{bmatrix} -2 \\ -1 \\ -\frac{1}{2} \end{bmatrix}$ **F.** $\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$ **G.** $\begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$ **H.** $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Answer: G. $\quad U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

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For the following two questions consider the following basis of $\mathbb{R}^2$:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

9. Find the coordinate vector $[5\mathbf{e}_2]_{\mathcal{B}}$:

A. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ **C.** $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ **D.** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **E.** $\begin{bmatrix} -5 \\ 10 \end{bmatrix}$ **F.** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **G.** $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **H.** $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$

Answer: F.

Because: $5\mathbf{e}_2 = 1 \cdot \mathbf{b}_1 + 2 \cdot \mathbf{b}_2$.

10. The matrix $[T]_{\mathcal{B}}$ of the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 - x_2 \\ 4x_1 + 3x_2 \end{bmatrix}$ relative to $\mathcal{B}$ is given by the matrix:

A. $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ **C.** $\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$ **D.** $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$ **F.** $\begin{bmatrix} -3 & -1 \\ 11 & 2 \end{bmatrix}$ **G.** $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ **H.** $\begin{bmatrix} -3 & 11 \\ -1 & 2 \end{bmatrix}$

Answer: B.

Because: $T(\mathbf{b}_1) = \begin{bmatrix} -3 \\ 11 \end{bmatrix} = \mathbf{b}_1 + 5\mathbf{b}_2$ and $T(\mathbf{b}_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0\mathbf{b}_1 + \mathbf{b}_2$.

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11. For which value of a is 3 an eigenvalue of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & a \end{bmatrix}$?

A. -3 **B.** -2 **C.** -1 **D.** 0 **E.** 1 **F.** 2 **G.** 3 **H.** 4

Answer: D.

$\det(A - 3I) = 4a$, so $E_3 = \text{Nul}(A - 3I) \neq \{\mathbf{0}\}$ if and only if $a = 0$.

12. Which of the following statements are **always** true for square matrices?

(I) If A is upper triangular $\Longrightarrow$ A is diagonalizable.

(II) If D is a diagonal matrix and $AP = PD \Longrightarrow A$ is diagonalizable.

A. Both statements are false.
B. Only (I) is true.
C. Only (II) is true.
D. Both statements are true.

Answer: A.

Counterexample for (I): $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Statement (II) can fail if P is not invertible.

Counterexample for (II): $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = I$, $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

13. Find a in the matrix A below such that A is diagonalizable:

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & a & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A. -8 **B.** -6 **C.** -4 **D.** -2 **E.** 0 **F.** 2 **G.** 4 **H.** 6

Answer: H.

The algebraic multiplicity of the eigenvalue 5 is equal to 2.

The geometric multiplicity of the eigenvalue 5 is equal to 2 if and only if $a = 6$.

The algebraic and geometric multiplicity of the other eigenvalues are equal to 1.

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For the following two questions consider the matrix $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.

14. The eigenvalues of A are given by

A. $-1 \pm \sqrt{3}i$
B. $1 \pm \sqrt{3}i$
C. $\pm 1 + \sqrt{3}i$
D. $1, \sqrt{3}$
E. $\sqrt{3} \pm i$
F. $1 \pm 3i$
G. $\sqrt{3}, \sqrt{3}$
H. $\sqrt{3} \pm 3i$

Answer: E.

In Lecture 16 we have seen that A has complex eigenvalues $\sqrt{3} \pm i$.

15. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(\mathbf{x}) = A\mathbf{x}$ is:

A. A rotation over an angle $\pi/6$ (counter-clockwise), followed by a scaling by factor 2
B. A rotation over an angle $\pi/6$ (counter-clockwise), followed by a scaling by factor 4

C. A rotation over an angle $\pi/3$ (counter-clockwise), followed by a scaling by factor 2
D. A rotation over an angle $\pi/3$ (counter-clockwise), followed by a scaling by factor 4
E. A rotation over an angle $\pi/6$ (clockwise), followed by a scaling by factor 2
F. A rotation over an angle $\pi/6$ (clockwise), followed by a scaling by factor 4
G. A rotation over an angle $\pi/3$ (clockwise), followed by a scaling by factor 2
H. A rotation over an angle $\pi/3$ (clockwise), followed by a scaling by factor 4

Answer: A.

The polar coordinates of $(\sqrt{3}, 1)$ are given by $(r, \varphi) = (2, \pi/6)$. In class (in the lecture on Complex eigenvalues and eigenvectors) we have seen that this implies that T is a rotation over an angle $\pi/6$ (counter-clockwise), followed by a scaling by factor 2.

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16. Consider the following statements for orthogonal $n \times n$ matrices U and V:

(I) $U + V$ is orthogonal.

(II) UV is orthogonal.

A. Both statements are false.
B. Only (I) is true.
C. Only (II) is true.
D. Both statements are true.

Answer: C.

Counterexample for (I): $U = I$, $V = -I$.

(II) is true since $(UV)^T UV = V^T U^T UV = V^T IV = V^T V = I$.

17. The distance from $\begin{bmatrix} 1 \\ 5 \\ -10 \end{bmatrix}$ to $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$ equals:

A. 6
B. $3\sqrt{5}$
C. 9
D. 45
E. 16
F. $3\sqrt{14}$
G. 126
H. 10

Answer: B.

The projection of $\mathbf{y}$ onto W is given by $\hat{\mathbf{y}} = \begin{bmatrix} 1 \\ 8 \\ -4 \end{bmatrix}$. The distance of $\mathbf{y}$ to W is, by definition, equal to the length of $\hat{\mathbf{y}} - \mathbf{y}$, i.e. of $\begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix}$, and this is equal to $\sqrt{45}$.

18. Applying Gram-Schmidt to the vectors $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$ we obtain, after rescaling, as third vector $\mathbf{v}_3$:

A. $\begin{bmatrix} 1 \\ -7 \\ 0 \\ 4 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 \\ -4 \\ 1 \\ 2 \end{bmatrix}$ **D.** $\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ **E.** $\begin{bmatrix} -3 \\ -11 \\ 8 \\ 4 \end{bmatrix}$ **F.** $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$ **G.** $\begin{bmatrix} -5 \\ 3 \\ 8 \\ -4 \end{bmatrix}$ **H.** $\begin{bmatrix} -7 \\ 17 \\ 8 \\ -12 \end{bmatrix}$

Answer: D.

Applying Gram-Schmidt yields, without rescaling, as third vector $\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$.

19. Determine the least-squares solution of the overdetermined system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$:

A. $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **D.** $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ **F.** $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **G.** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **H.** There are no least-squares solutions

Answer: C.

The normal equations $A^T A\hat{x} = A^T b$ are given by:

$$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \hat{x} = \begin{bmatrix} -4 \\ -12 \end{bmatrix}$$

The unique solution is therefore given by $\hat{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

END OF PART I.

GO TO PART II: TRUE/FALSE QUESTIONS

Resit *Linear Algebra* (CSE1205): True/False Questions
July 4, 2019, 13:30-16:30

- In the following questions you are asked to decide whether the statements are true or false.
- If you think the statement is true, explain clearly why.
- Give a counterexample (with explanation) if you think the statement is false.
- **Simply writing true or false is not enough.**
- Credits: **4 points** for every True/False questions.

20. If A is a 3×3 matrix such that $A\mathbf{x} = 0$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for each $\mathbf{b} \in \mathbb{R}^3$.

Answer: **FALSE**.

A counterexample is given by: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, but $A\mathbf{x} = \mathbf{b}$ has NO solutions.

(But any non-invertible matrix A actually will do the trick, but one has to choose $\mathbf{b}$ suitably).

21. If $\mathbf{v}$ is an eigenvector of the matrices A and B, then $\mathbf{v}$ is also an eigenvector of AB.

Answer: **TRUE**.

Let us denote the corresponding eigenvalues of A and B by λ and μ, respectively:

$$A\mathbf{v} = \lambda\mathbf{v}, \qquad B\mathbf{v} = \mu\mathbf{v}$$

Observe that in general $\lambda \neq \mu$.

Therefore:

$$AB\mathbf{v} = A(\mu\mathbf{v}) = \mu A\mathbf{v} = \mu\lambda\mathbf{v}$$

So $\mathbf{v}$ is an eigenvector of AB with eigenvalue $\lambda\mu$.

22. If the unit vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$ are orthogonal, then the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are also orthogonal.

Answer: **TRUE**.

The vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal if and only if their dot product is equal to 0. .

Taking the dot product of $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$:

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}. \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}. \\ &= 1 - 1 = 0.\end{aligned}$$

So $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are indeed orthogonal.