Exam Linear Algebra (TI1206M)
May 28 2015, 6:30 pm – 9:30 pm

Important: Do not only give the final answers (unless explicitly asked); include all the relevant calculations and arguments to arrive at the answers. Write clearly and formulate in correct English sentences. The use of a calculator is not permitted.
Credits: 7 + 11 + 7 + 10 + 7

1. In this exercise, we examine the following matrix and vectors:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & h \\ -1 & -3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{en} \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

a. For which value(s) of h is $\mathbf{b}$ an element of Col A? Consistent system

b. For which value (s) of h is $\mathbf{u}$ an element of Nul A? Au = 0 = nullspace

c. For which value(s) of h is the linear transformation T, defined by $T(\mathbf{x}) = A\mathbf{x}$, one-to-one? Start your answer with a definition of 'one-to-one'.

2. Use scrap paper to solve this exercise and **only give final answers**.

a. Find the LU-factorization of the matrix $\begin{bmatrix} 2 & 3 & 2 \\ -2 & 2 & -4 \\ 1 & -1 & 5 \end{bmatrix}$.

b. Determine the determinant of the matrix $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Tip: use two approaches to calculate the determinant so you can verify that you get the same answer both times.

c. Give the distance and angle between the vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}$.

Give **exact** answers (for example: not 1.732 but $\sqrt{3}$); you may leave an arcsin or arccos for 'unusual' angles.

d. Find the least-squares solution of the system $\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 - x_3 = -7 \\ x_2 + x_3 = 1 \\ x_1 - x_2 + x_3 = -2 \end{cases}$

(As a check: the solution is integer!)

3. This exercise focuses on the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & -1 \\ 9 & 3 & -5 \end{bmatrix}$.

a. Check whether the vectors $\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 8 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ are eigenvectors of A.

b. Show that $\lambda = -2$ is an eigenvalue of A and give a basis for the eigenspace for this eigenvalue.

c. Give the characteristic equation of A *without first finding the determinant*. Also explain (in words) how you arrived at the answer!

4. Given are the following four vectors in $\mathbb{R}^4$:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and } \mathbf{y} = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 1 \end{bmatrix}$$

Also let $W = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $A = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,]$.

a. Find an orthogonal basis for W.

b. Give a basis for the orthogonal complement of W.

c. Write the vector $\mathbf{y}$ as a sum of a vector in W and a vector in $W^\perp$.

d. Find the distance from $\mathbf{y}$ to W. Explain (in words) the logic behind it it!

5. Give the definitions of the underlined terms below, and prove the propositions:

a. If A is an invertible matrix, then $2A$ is also an invertible matrix.

b. If V and W are subspaces of $\mathbb{R}^n$, then the intersection $V \cap W$ is also a subspace. (The intersection contains exactly those vectors that lie in both subspaces.)

c. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set of vectors in $\mathbb{R}^n$ and $\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$, then the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is orthogonal.

SOLUTIONS

1a This question is equivalent to: for which h is the system $A\mathbf{x} = \mathbf{b}$ consistent? So:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 4 & h & 3 \\ -1 & -3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 0 & h+4 & 1 \\ 0 & -1 & -1 & 3 \end{array}\right]$$

The system is only consistent if $\boxed{h \neq -4}$.

1b $\mathbf{u} \in \operatorname{Nul} A$ if $A\mathbf{u} = \mathbf{0}$.

Since $A\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, this is <u>not the case for **any** h.</u>

1c $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if for all $\mathbf{y} \in \mathbb{R}^m$ the equation $T(\mathbf{x}) = \mathbf{y}$ <u>has at most one</u> solution. This is the case if, and only if, every column of A is a pivot column. The last matrix from the answer to part **a.** shows that this occurs when $\boxed{h \neq -4}$.

2a $$\begin{bmatrix} 2 & 3 & 2 \\ -2 & 2 & -4 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 3 \end{bmatrix}.$$

2b Step 1: Reduce with row 4; step 2: reduce with row 3:

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

Next expand row 1 and then column 1:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$$

2c The distance: $\operatorname{dist}(\mathbf{v}, \mathbf{w}) = ||\mathbf{v} - \mathbf{w}|| = \sqrt{1^2 + 1^2 + (-3)^2 + (-3)^2} = \sqrt{20} = 2\sqrt{5}$.

The angle φ: $\cos\varphi = \dfrac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \cdot ||\mathbf{w}||} = \dfrac{-4}{\sqrt{4} \cdot \sqrt{8}} = -\dfrac{1}{\sqrt{2}}$, and the only angle (between 0 and π) with this cosine is: $\boxed{\varphi = \frac{3}{4}\pi}$.

2d Question: the least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ -7 \\ 1 \\ -2 \end{bmatrix} = \mathbf{b}$.

That is: the solution of $A^T A\mathbf{x} = A^T\mathbf{b}$:

$$[\, A^T A \mid A^T\mathbf{b} \,] = \left[\begin{array}{ccc|c} 3 & 0 & 1 & -3 \\ 0 & 3 & 1 & 9 \\ 1 & 1 & 4 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & -3 \\ 0 & 3 & 1 & 9 \\ 0 & 0 & 4 - \frac{2}{3} & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & -3 \\ 0 & 3 & 1 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

A little more reduction yields: $\hat{\mathbf{x}} = [-2 \;\; 2 \;\; 3]^T$.

3a $A\mathbf{v}_1 = \begin{bmatrix} -10 \\ 14 \\ -16 \end{bmatrix} = (-2)\mathbf{v}_1, \quad A\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = (-1)\mathbf{v}_1$, so $\mathbf{v}_1$ is an eigenvector for $\lambda_1 = -2$, and $\mathbf{v}_2$ is an eigenvector for $\lambda_2 = -1$.

3b The fact that (-2) is an eigenvalue was shown in part **a**. To find a basis for the eigenspace, solving $(A - (-2)I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 1-(-2) & 1 & -1 & 0 \\ 3 & -1-(-2) & -1 & 0 \\ 9 & 3 & -5-(-2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 3 & 1 & -1 & 0 \\ 9 & 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields two independent eigenvectors, for instance $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for the eigenspace.

3c From before we see that: A has the eigenvalue $\lambda_1 = -1$, and the eigenvalue $\lambda_1 = -2$ with an algebraic multiplicity of 2. The characteristic equation is divisible by $\lambda - (-1)$ and by $(\lambda - (-2))^2$. Since the polynomial is of the third degree and the coefficient of λ^3 equals $(-1)^3$, it follows that the characteristic equation is equal to $-(\lambda + 1)(\lambda + 2)^2$.

4a Gram-Schmidt:

$$\mathbf{b}_1 = \mathbf{a}_1, \quad \mathbf{b}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -1 \\ -1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -3 \\ -1 \end{bmatrix}$$

We continue with: $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -1 \end{bmatrix}$. Then:

$$\mathbf{b}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2}\mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{5}{15}\begin{bmatrix} 2 \\ 1 \\ -3 \\ -1 \end{bmatrix} = \ldots = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Apparently, $\mathbf{a}_3$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$ (looking back, indeed: $\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_2$)
Possible answer: $\{\mathbf{b}_1, \mathbf{b}_2\}$. (Note: the zero vector does not belong in the basis!)

4b We're looking for a basis for $(\text{span}\{\mathbf{a}_1, \mathbf{a}_2\})^\perp = \text{Nul}\left([\mathbf{a}_1\ \mathbf{a}_2]^T\right)$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \end{array}\right]$$

The pivots are in the first and third columns (crazy! ;-).

Solving the 'system' (which immediately gives us a basis): $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

4c The question is: to write $\mathbf{y}$ as $\mathbf{w}+\mathbf{v}$ with $\mathbf{w} \in W$ and $\mathbf{v} \in W^{\perp}$.
For $\mathbf{w}$ you need to find the (orthogonal) projection of $\mathbf{y}$ on W, which is quickly achieved using the orthogonal basis from part **a.**:

$$\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{b}_1}{\mathbf{b}_1\cdot\mathbf{b}_1}\mathbf{b}_1 + \frac{\mathbf{y}\cdot\mathbf{b}_2}{\mathbf{b}_2\cdot\mathbf{b}_2}\mathbf{b}_2 = \frac{6}{3}\begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + \frac{-15}{15}\begin{bmatrix}2\\1\\-3\\-1\end{bmatrix} = \begin{bmatrix}0\\-3\\3\\3\end{bmatrix}$$

and then $\mathbf{v} = \mathbf{y} - \mathbf{w} = \begin{bmatrix}3\\1\\3\\-2\end{bmatrix}$

4d The distance from $\mathbf{y}$ to W is the (orthogonal) distance from $\mathbf{y}$ to W, which exactly equals the distance from $\mathbf{y}$ to $\hat{\mathbf{y}}$ and $\hat{\mathbf{y}} = \mathbf{w}$:

$$\text{dist}(\mathbf{y}, W) = \text{dist}(\mathbf{y}, \mathbf{w}) = ||\mathbf{y}-\mathbf{w}|| = ||\mathbf{v}|| = \sqrt{3^2+1^2+3^2+(-2)^2} = \sqrt{23}$$

5a A matrix A is invertible if there exists a matrix C so that $AC = CA = I$.
Suppose A is invertible, and take $C = A^{-1}$. Then $(2A)\cdot(\frac{1}{2}C) = 2\cdot\frac{1}{2}\cdot AC = 2\cdot\frac{1}{2}I = I$, and $(\frac{1}{2}C)\cdot(2A) = \frac{1}{2}\cdot 2CA = \frac{1}{2}\cdot 2I = I$, so $2A$ is invertible and has as inverse $\frac{1}{2}C$ $(= \frac{1}{2}A^{-1})$.

5b H is a subspace in $\mathbb{R}^n$ if

(i) H is not empty. (Also fine: the zero vector is an element of H).
(ii) If $\mathbf{u}$ and $\mathbf{v}$ are in H then $\mathbf{u}+\mathbf{v}$ is also in H.
(iii) If $\mathbf{u} \in H$ and c is a real number, then also $c\mathbf{u} \in H$.

Say V and W are subspaces.
They both contain the zero vector, so $\mathbf{0} \in V \cap W$.
For (ii): if $\mathbf{u}$ and $\mathbf{v}$ lie in $V \cap W$, then $\mathbf{u}$ and $\mathbf{v}$ lie in V as well as in W, and since V and W satisfy (ii), $\mathbf{u}+\mathbf{v}$ lies in V as well as in W, so in $V \cap W$.
For (iii): if $\mathbf{u}$ is in $V \cap W$ and c is a number, then $c\mathbf{u}$ lies in V and also in W, because V and W both satisfy (iii), so $c\mathbf{u}$ lies in $V \cap W$.

5c The definition states: $\mathbf{u}_i \cdot \mathbf{u}_i = 1$, and $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$.
So $\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{u}_1+\mathbf{u}_2)\cdot(\mathbf{u}_1-\mathbf{u}_2) = \mathbf{u}_1\cdot\mathbf{u}_1 - \mathbf{u}_1\cdot\mathbf{u}_2 + \mathbf{u}_2\cdot\mathbf{u}_1 - \mathbf{u}_2\cdot\mathbf{u}_2 = 1-0+0-1 = 0$, and thus $\mathbf{v}_1 \perp \mathbf{v}_2$. $\square$