Linear Algebra Resit $\mathbb{TI}1206\mathbb{M}$
April 13, 2016, 09.00 – 12.00 uur

Important: Do not only give the final answers (unless explicitly asked): include all the relevant calculations and arguments to arrive at the answers. Write clearly and formulate in correct English. The uses of a calculator is not permitted.
Credits: 1. **7** pts; 2. **7** pts; 3. **3** pts; 4. **7** pts; 5. **8** pts; 6. **8** pts.

---

1. Given are the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & \alpha & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

a. For which α is $\mathbf{b}$ an element of the column space of A?

b. Calculate all the values of α for which the columns of A are dependent.

c. Give a basis for $\operatorname{Nul} A$ for the case that $\alpha = -1$.

d. For which value(s) of α is $\mathbf{b}$ an element of the null space of A?

2. Given are two matrices and a vector:

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & 6 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 7 \\ 7 \\ 7 \\ -7 \\ -7 \end{bmatrix}$$

a. Determine the determinant of A.
(Tip: first subtract row five from the other rows.)

b. Also given is that $AB = 14I$. (Which you don't have to check!)
Use these givens to find A^{-1}.

c. Solve: $A\mathbf{x} = \mathbf{p}$. (Tip: use part **b.**)

3. Calculate the least-squares line $y = \alpha + \beta x$ belonging to the points (0,0), (1,4), (2,5), (3,8), (4,8).

4. Given is the matrix $A = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

a. Show that $\lambda = -1$ is an eigenvalue of A.

b. Check that $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A.

c. Is A diagonalizable? If not, explain why; if so, give a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

5. **a.** Use the Gram-Schmidt method to find an orthonormal basis for the subspace S of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix},$$

b. Describe (in words!) a method to find the orthogonal complement of S, and find such a basis.

c. Calculate the orthogonal projection of the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \\ 2 \end{bmatrix}$ on S.

6. For each of the following statements, give a proof if it's true, and a counterexample if it's false:

In the first two parts, A is some 4×7 matrix.

a. The system $A\mathbf{x} = \mathbf{b}$ (always) has infinitely many solutions for every $\mathbf{b}$ in $\mathbb{R}^4$

b. The system $A\mathbf{x} = \mathbf{0}$, the zero vector in $\mathbb{R}^4$, always has at least three *independent* solutions.

c. If an $n \times n$ matrix A is diagonalizable then $2A + I$ is also diagonizable.

d. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are an orthogonal set of vectors in $\mathbb{R}^4$, then the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, 2\mathbf{u}_3\}$ is also orthogonal.

SOLUTIONS

1a Subtract the first row from the second, and the fourth from the third.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & \alpha & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & \alpha-1 & 0 & 2 \\ 0 & 1 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2+2(\alpha-1) \\ 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

And it's clear that $A\mathbf{x} = \mathbf{b}$ only has solutions if $2 + 2(\alpha - 1) = 0$, so when $\boxed{\alpha = 0}$.

1b You can see it right away: $\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_4$, independent of α, the columns are *always* dependent. You can also look back at part **a.** and note that A (for every value of α) has exactly three pivots; so not every column is a pivot; so the columns are *always* dependent.

1c In **a.** we found that $A \sim E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the null spaces of A and E are equal. There are three pivots, so $\dim \text{Nul}(E) = 1$ and the relationship mentioned in **b.** (or taking a look at matrix E) shows you that $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\text{Nul}(A)$.

1d $A\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ \alpha - 2 \\ -1 \end{bmatrix}$ is equal to the zero vector for no values of α, so $\mathbf{b} \in \text{Nul}\, A$ for no values of α.

2a Many roads lead to Rome. Following the top (why wouldn't you): subtract the bottom row from the other rows, and then add the first four columns to the fifth column (you'll see why after the first step!):

$$\begin{vmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 7 \end{vmatrix} = |L|$$

And the determinant of a triangle matrix is simply the product of the elements on the diagonal: $|A| = |L| = 2^4 \cdot 7 = 112$

2b From $AB = 14I$ follows $\frac{1}{14}AB = A(\frac{1}{14}B) = I$, and that pretty much shows that $\frac{1}{14}B$ is the inverse of A (because one of the invertible matrix rules is that, for a square matrix A: if $AB = I$ then $B = A^{-1}$).

2c You just need one matrix multiplication: we know that A is invertible, so

$$A\mathbf{x} = \mathbf{p} \Longleftrightarrow \mathbf{x} = A^{-1}\mathbf{p} = \frac{1}{14}\begin{bmatrix} 6 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \\ 7 \\ -7 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ -4 \\ -4 \end{bmatrix}$$

3 To find α and β : calculate the least-squares solution of the system

$$\alpha\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 5 \\ 8 \\ 8 \end{bmatrix}$$

Normal equations: $\left[\begin{array}{cc|c} 5 & 10 & 25 \\ 10 & 30 & 70 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$, gives $\alpha = 1,\ \beta = 2$.

4a To check: $A\mathbf{x} = (-1)\mathbf{x}$, so $(A - (-1)I)\mathbf{x} = \mathbf{0}$, has a solution $\mathbf{x} \neq \mathbf{0}$.

$$[A - (-1)I \,|\, \mathbf{0}] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It's clear there are two independent solutions to this system.

4b

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Shown here is that the matrix has 2 as eigenvalue.

4c From **a.** it follows that there are two independent eigenvectors for eigenvalue -1, and part **b.** delivers a third (automatically independent) eigenvector at e.w. 2. These three form a basis of eigenvectors, and that's one of the characteristics of diagonalizability.

$A = PDP^{-1}$, for (e.g.) $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(The first two columns of P are eigenvectors for eigenvalue $\lambda = -1$, following from the reduced matrix from part **a**.)

5a

$$\begin{aligned} \mathbf{b}_1 &= \mathbf{a}_1. \\ \mathbf{b}_2 &= \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = [0\,3\;1\;\; -1]^T. \\ \mathbf{b}_3 &= [1\;\; -3\;0\;2]^T - \frac{3}{3}[1\;0\;1\;1]^T - \frac{-11}{11}[0\;3\;1\;\; -1]^T = [0\;0\;0\;0]^T. \end{aligned}$$

It's clear that $\mathbf{a}_3$ is dependent on $\mathbf{a}_1$ and $\mathbf{a}_2$.

Normalization gives the ortho<u>normal</u> basis

$$\{\mathbf{q}_1, \mathbf{q}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{11}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \quad \right\}$$

5b The orthogonal complement consists of all vectors that are orthogonal to $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ (or $\mathbf{b}_1$ en $\mathbf{b}_2$). If we work with the $\mathbf{b}$-basis then those are the solutions of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right]$$

This system has the solutions $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$

And an obvious basis is (alongside, of course, an infinite amount of others): $\{\mathbf{u}_1, \mathbf{u}_2\}$.

5c $\mathbf{v} = 2\mathbf{a}_1 + \mathbf{a}_2$ lies in S, so $\text{proj}_S(\mathbf{v}) = \mathbf{v}$.

If you didn't see this, you can still use the non-normalzed **orthogonal** basis

$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}$ from part **a.**:

$$\hat{\mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{9}{3}\mathbf{u}_1 + \frac{11}{11}\mathbf{u}_2 = \begin{bmatrix} 3 \\ 3 \\ 4 \\ 2 \end{bmatrix}$$

6a A has four rows and seven columns. The proposition is **false**: in general, the seven columns can span the whole $\mathbb{R}^4$ but they don't <u>have</u> to. Silly example: take the zero matrix for A !

6b This is true: reducing A gives an echelon matrix with <u>at most</u> four pivots (because there are only four rows); that means that there are <u>at least</u> three columns without pivots, and there are at least three variables in the solution to $A\mathbf{x} = \mathbf{0}$, which implies that there are at least three independent solutions.

6c This is true: take $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ as a basis of eigenvectors for the eigenvalues $d_1, d_2, ..., d_n$, which is then also a basis for $2A + I$, since $(2A + I)\mathbf{v}_i = 2A\mathbf{v}_i + I\mathbf{v}_i = 2d_i\mathbf{v} + \mathbf{v}_i = (2d_i + 1)\mathbf{v}_i$.

(A bit) shorter: $A = PDP^{-1} \Rightarrow 2A+I = 2PDP^{-1}+I = 2PDP^{-1}+PIP^{-1} = P(2D+I)P^{-1}$, and the final step gives a diagonalization for $2A+I$, because $2D+I$ is another diagonal matrix.

6d This is true: the given is equivalent to $\mathbf{u}_i \cdot \mathbf{u}_i = 1$, and $\mathbf{u}_i \cdot \mathbf{u}_j = 0$, if $i \neq j$.

To show: $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{u}_1 + \mathbf{u}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{u}_1 \cdot \mathbf{u}_1 - \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_2 \cdot \mathbf{u}_1 - \mathbf{u}_2 \cdot \mathbf{u}_2 = 1 - 0 + 0 - 1 = 0,$

$\mathbf{v}_1 \cdot \mathbf{v}_3 = (\mathbf{u}_1 + \mathbf{u}_2) \cdot (2\mathbf{u}_3) = 2\mathbf{u}_1 \cdot \mathbf{u}_3 + 2\mathbf{u}_2 \cdot \mathbf{u}_3 = 2 \cdot 0 + 2 \cdot 0 = 0,$

and, analogously, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

NORMEN

1a 3 pt. foute conclusie door niet ‘doorvegen’: **0** pt.

1b 1 pt.

1c 2 pt. Alleen ‘algemene opl.’ gegeven: **0.5** pt eraf.

1d 1 pt.

2a 3 pt. Alternatieve aanpak: per rekenfout **1.5 pt** eraf.

2b 2 pt.

2c 2 pt.

3 3 pt. **2** voor normaalvergelijkingen

4a 2 pt.

4b 2 pt.

4c 3 pt.

5a 3 pt. nulvector in basis opgenomen: **1** eraf; niet genormeerd: ook **1** eraf.

5b 3 pt. Geen goed geformuleerde uitleg (of uitleg die niet past bij de uiteindelijke berekening) **1** pt. eraf;

5c 2 pt. Projectieformule voor niet-orthogonale basis gebruikt: **0** pt.

6 8 pt. **2** pt per onderdeel
ad **a.**: indien alleen opgemerkt ‘stelsel kan strijdig zijn’: max **1** pt.
ad **b.**: indien beweerd ’A heeft vier pivots’ **1** pt. eraf

$$\textbf{cijfer} := \frac{\textbf{totaal} + 4}{4.4}, \textbf{ afgerond op halven.}$$