Tentamen Lineaire Algebra $\mathbb{TI}1206\mathbb{M}$
19 mei 2016, 18.30 – 21.30 uur

Important: do not only give the final answers (unless explicitly asked): include all the relevant calculations and arguments to arrive at the answers. Write clearly and formulate in correct English. The uses of a calculator is not permitted.
Credits: ex.1: 7; ex.2: 9; ex.3: 8; ex.4: 5 (+2); ex.5: 8.

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1. Given is the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & \alpha \\ 3 & 1 & \alpha & 1 \\ 4 & 3 & 4 & 4 \end{bmatrix}$

a. Carefully convert the matrix to reduced echelon form. (This may yield different results for different values of α. Specifically note these.)

b. Give a basis for the column space for $\alpha = 2$.

c. Give a basis for the null space for $\alpha = 1$.

2. For this question, only the answers are required!!
(Hint: for **a.** and **b.** your answers are easy to check!)

a. Calculate the inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & 2 \end{bmatrix}$.

b. Find the LU-factorization of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ -4 & 0 & -3 \\ 2 & -1 & 5 \end{bmatrix}$

c. Give a matrix P and a diagonal matrix D such that $\begin{bmatrix} 2 & -2 \\ 3 & -5 \end{bmatrix} = PDP^{-1}$.
(No need to calculate P^{-1} .)

P = Eigenspace, D = Eigenvalues

d. Calculate the least-squares solution of the system

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -13 \end{bmatrix}.$$

(The answer is an integer.)

3. Given are the following matrix A and the vectors $\mathbf{u}$ and $\mathbf{y}$:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ -1 \end{bmatrix}.$$

a. Give the definition of a subspace W in $\mathbf{R}^n$.

b. Show that $\{\mathbf{u}\}$ is a basis for $(\mathrm{Col}\,A)^{\perp}$.

c. Find the projection of $\mathbf{y}$ on $\mathrm{Col}\,A$.

d. Give the ('standard') matrix of the projection on $\mathrm{Col}\,A$.

4. Given is the matrix $C = \begin{bmatrix} -3 & 6 & 8 \\ -1 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix}$ and the vector $\mathbf{w} = \begin{bmatrix} 2i \\ 3+i \\ -2 \end{bmatrix}$.

The characteristic equation is a gift: $p(\lambda) = -(\lambda+1)(\lambda^2+1)$.

a. Give all eigenvalues of C (including complex ones), and for every real eigenvalue also a basis of eigenvectors.

b. Check if $\mathbf{w}$ is an eigenvector of C. If so: give each an eigenvalue.

c. If we diagonalize C complexely, by writing it as PDP^{-1}, with P and D possibly complex matrices, what does the diagonal matrix D look like?

d. [bonus] Find C^{101}. (Also give a clear explanation.)

5. For each of the following statements, indicate whether it is true or false. Also: **provide clear explanations** in **correct** English. Complete sentences!

a. If V and W are subspaces of $\mathbb{R}^n$, then the intersection $V \cap W$ is also a subspace of $\mathbb{R}^n$. (In **3.a.** you already definded 'subspace'!)

b. [continuation of part **a.**] If $V = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $W = \mathrm{span}\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ are subspaces of $\mathbb{R}^n$, with $\mathbf{a}_1, \ldots, \mathbf{a}_5$ five different vectors, all unequal to $\mathbf{0}$, then $\{\mathbf{a}_3\}$ is a basis for the intersection $V \cap W$.

c. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set of vectors in $\mathbb{R}^5$, then the set of vectors $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, 2\mathbf{u}_3\}$ is also orthogonal.

d. For every pair of $n \times n$ matrices it is true that: if λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.

e. For every pair of matrices (for which the given product exists), it holds that: each vector that lies in the null space of B, also lies in the null space of AB.

$\mathcal{END}$

SOLUTIONS

1a

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 1 & 2 & 1 & \alpha \\ 3 & 1 & \alpha & 1 \\ 4 & 3 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 0 & \alpha-1 \\ 0 & -2 & \alpha-3 & -2 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2-\alpha \\ 0 & 1 & 0 & \alpha-1 \\ 0 & 0 & \alpha-3 & 2\alpha-4 \\ 0 & 0 & 0 & \alpha-1 \end{bmatrix}$$

If $\alpha \neq 1, 3$: four pivots, so reduced echelon form is I_4 (identity matrix).
If $\alpha = 3$:

$$\begin{bmatrix} 1 & 0 & 1 & 2-3 \\ 0 & 1 & 0 & 3-1 \\ 0 & 0 & 3-3 & 6-4 \\ 0 & 0 & 0 & 3-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{-2} \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And finally, if $\alpha = 1$:

$$\begin{bmatrix} 1 & 0 & 1 & 2-1 \\ 0 & 1 & 0 & 1-1 \\ 0 & 0 & 1-3 & 2-4 \\ 0 & 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-2} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1b From **a.** follows: for $\alpha = 2$ the matrix has four pivots. The column space spans the whole $\mathbb{R}^4$, so a possible basis would be the four columns of A, or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, the 'standard' basis.

1c For $\alpha = 1$, we calculated the reduced echelon form E of A in part **a.**, and $\operatorname{Nul} A = \operatorname{Nul} E$. For the solution of $E\mathbf{x} = \mathbf{0}$ you can take x_4 as a free variable.

As basis for $\operatorname{Nul} A$ you can take $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$. (Hmm, pretty easy, actually.)

2a

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6\frac{1}{2} & -3\frac{1}{2} & 1\frac{1}{2} \\ 0 & 1 & 0 & -2\frac{1}{2} & 1\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

Scaling the bottom row yields the inverse: $\begin{bmatrix} 6\frac{1}{2} & -3\frac{1}{2} & 1\frac{1}{2} \\ -2\frac{1}{2} & 1\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 13 & -7 & 3 \\ -5 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$

2b $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

2c The most obvious answers:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \text{ bij } D = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}, \quad \text{of } P = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \text{ bij } D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$$

2d $\hat{\mathbf{x}} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

3a See textbook, § 2.8.

3b Approach 1: $A^T\mathbf{u} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 = \mathbf{0}$, so $\mathbf{u}$ is orthogonal to all columns of A, and so is in $(\mathrm{Col}\,A)^\perp$.
If we can show that $\dim(\mathrm{Col}\,A)^\perp = 1$, or, that $\dim(\mathrm{Col}\,A) = 3$, then we're there.

Let's reduce: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
shows there are three columns, so $\dim(\mathrm{Col}\,A) = 3$.

Approach 2: you can also 'just' solve $A^T\mathbf{x} = \mathbf{0}$ and check that you get one independent solution that's a multiple of $\mathbf{u}$.

3c We can use part **b**.)!
Lemma: $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, for which $\mathbf{z}$ is the projection of $\mathbf{y}$ on $(\mathrm{Col}\,A)^\perp$.
We use the given basis to find the second projection $\{\mathbf{u}\}$:

$$\mathbf{z} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{16}{4}\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -4 \end{bmatrix} \implies \hat{\mathbf{y}} = \mathbf{y} - \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Note: you **<u>can't</u>** use the 'standard projection formula' for the non-orthogonal columns of A.

3d You can reuse the steps from part **c.** for any vector $\mathbf{x}$.

'Abstractly': $\hat{\mathbf{x}} = \mathbf{x} - \dfrac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \mathbf{x} - \dfrac{\mathbf{u}^T\mathbf{x}}{4}\mathbf{u} = \mathrm{I}\,\mathbf{x} - \tfrac{1}{4}\mathbf{u}\mathbf{u}^T\mathbf{x} = \left(\mathrm{I} - \tfrac{1}{4}\mathbf{u}\mathbf{u}^T\right)\mathbf{x}$,
so the projection matrix becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

Less abstract: 'immediately' fill in $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ en $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ in de uitdrukking $\mathbf{x} - \dfrac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$
in, en herleid dit tot de vorm $P\,\mathbf{x}$.

Also possible: column by column: $\begin{bmatrix} P(\mathbf{e}_1) & P(\mathbf{e}_2) & P(\mathbf{e}_3) & P(\mathbf{e}_4) \end{bmatrix}$.

4a Easy: the eigenvalues are the zeros of the characteristic equation: $\lambda_1 = -1,\ \lambda_{2,3} = \pm i$.
Eigenvectors for λ_1:

$$\left[\, C-(-1)\mathrm{I} \;\middle|\; \mathbf{0} \right] = \left[\begin{array}{ccc|c} -2 & 6 & 8 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yields eigenvectors $\mathbf{v} = c \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. A basis could be $\left\{ \begin{bmatrix} 30 \\ 30 \\ -15 \end{bmatrix} \right\}$.

4b $C\mathbf{w} = \begin{bmatrix} 2 \\ 1-3i \\ 2i \end{bmatrix}$ which is equal to $(-i)\,\mathbf{w}$.
So, $\mathbf{w}$ is an eigenvector for $\lambda = -i$.

4c D has three eigenvalues -1 and $\pm i$ on the diagonal. The order doesn't matter.

4d Take, for example, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & +i & 0 \\ 0 & 0 & -i \end{bmatrix}$, and a 'fitting' P (such that $C = PDP^{-1}$).
From the fact that $(\pm i)^4 = (-1)^2 = 1$ and also $(-1)^4 = 1$, it follows that $D^4 = I$,
$D^{100} = (D^4)^{25} = I$, en $D^{101} = D \cdot D^{100} = D$.
Then: $C^{101} = (PDP^{-1})^{101} = PD^{101}P^{-1} = PDP^{-1} = C$.

5a This is true: V and W both contain the zero vector, so $V \cap W$ also does; so $V \cap W \neq \emptyset$.
If $\mathbf{u}$ and $\mathbf{v}$ are in $V \cap W$, then they are in V as well as W,
and since V and W are subspaces, $\mathbf{u}+\mathbf{v}$ is in V as well as W, so in $V \cap W$.

Finally: if $\mathbf{u}$ is in $V \cap W$ and $c \in \mathbb{R}$, then $\mathbf{u}$ is in (subspace) V, so also $c\mathbf{u} \in V$; if $\mathbf{u}$ is in (subspace) W, then also $c\mathbf{u} \in W$, which together shows that $c\mathbf{u} \in V \cap W$.

$V \cap W$ meets all the requirements for a subspace. Q.E.D.

5b False: take, for example, $\{\mathbf{a}_1, \ldots, \mathbf{a}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.
this gives $V = W = \mathbb{R}^3$, so also $V \cap W = \mathbb{R}^3$.

5c This is false: $(\mathbf{u}_1+\mathbf{u}_2)\cdot(\mathbf{u}_1-\mathbf{u}_2) = \ldots = \mathbf{u}_1\cdot\mathbf{u}_1 - \mathbf{u}_2\cdot\mathbf{u}_2$, which does not have to be equal if $\mathbf{u}_1$ and $\mathbf{u}_2$ are 'just' orthogonal (for orthonormal vectors it would hold).

Take, for example $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

5d This is **false**. Take, for exaple, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then A has eigenvalue $\lambda = 1$
and B has eigenvalue $\mu = 5$, but $AB = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}$ doesn't have the eigenvalue $1 \cdot 5$.

5e This is **true**. If $\mathbf{v} \in \text{Nul}(B)$, then $B\mathbf{v} = \mathbf{0}$, so automatically also $AB\mathbf{v} = A(B\mathbf{v}) = A\,\mathbf{0} = \mathbf{0}$, so $\mathbf{v} \in \text{Nul}(AB)$. This shows that always $\text{Nul}(B) \subset \text{Nul}(AB)$.

NORMEN

1a 4 pt. Indien geen onderscheid gemaakt naar α: maximaal **2** pt.

1b 1 pt. geen aftrek indien gegeven $\text{Col}\,A = \text{span}\{\dots\}$

1c 2 pt. **0.5** aftrek indien niet expliciet als basis gegeven.

2a 2 pt.

2b 2 pt. **1** pt. voor L, **1** pt. voor U

2c 3 pt. **1** pt. voor D, **2** pt. voor P

2d 2 pt.

3a 1 pt.

3b 2 pt. **1** pt. voor checken $A^T\mathbf{u} = \mathbf{0}$, **1** pt. voor controleren dimensie.
('t Kan natuurlijk ook door domweg op te lossen $A^T\mathbf{x} = \mathbf{0}$.)

3c 3 pt. Legio mogelijkheden.
Foute formule toegepast bij niet-orthogonale basis: **2** punten **eraf**.
Dit geeft overigens het 'foute' antwoord $[0\ 0\ 2\ 2]^T$.

3d 2 pt. N.b.v.z.

4a 2.5 pt.

4b 1.5 pt.

4c 1 pt.

4d 2 pt.

5a 2.5 pt. Checken $V \cap W \neq \emptyset$: **0.5** pt;
voor geslotenheid onder som en veelvoud: $2\times$ **1** pt.

5b 1 pt.

5c 1 pt.

5d 2 pt.

5e 2 pt.