Exam WI4410 Advanced Discrete Optimization

August 23, 2017, 13:30–16:30

The exam consists of 6 questions. In total you can obtain 60 points. Your grade is calculated by dividing the number of points you obtained by 6. You may use a non-graphical calculator during the exam. Using a graphical calculator, notes, phone, smart-watch, etc. is **not** permitted. The total number of pages of this exam is 8. Good luck!

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1. (a) (4 points) Consider the pure integer linear set

$$S = \{\boldsymbol{x} \in \mathbb{Z}^2_+ \mid x_1 + x_2 \leq 4,\ -5x_1 + 11x_2 \geq 0\}$$

and the split disjunction

$$x_2 \geq 2 \quad \text{or} \quad x_2 \leq 1\,.$$

Derive a split inequality for this set given the split disjunction. You may use a graphical representation of the problem as support in your derivation. State the inequality that you have derived mathematically, and give a short explanation of how you derived it.

Solution: Take the intersection of the constraint $x_1 + x_2 \leq 4$ with the inequality $x_2 \geq 2$ of the split disjunction. This intersection is in the point $(x_1,\ x_2)^T = (2,\ 2)$. Similarly, take the intersection of the constraint $-5x_1 + 11x_2 \geq 0$ with the inequality $x_2 \leq 1$ of the split disjunction. This intersection is in the point $(x_1,\ x_2)^T = (11/5,\ 1)$. The line that goes through these points is: $5x_1 + x_2 = 12$, and it yields the constraint

$$5x_1 + x_2 \leq 12$$

as a valid split cut.

(b) (3 points) Gomory's mixed-integer cut (GMIC) for the mixed-integer set

$$S = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ \mid \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b\}$$

is as follows:

$$\sum_{f_j \leq f} f_j x_j + \sum_{f_j > f} \frac{f(1 - f_j)}{1 - f} x_j + \sum_{a_{ij} \geq 0} a_{ij} y_j - \sum_{a_{ij} < 0} \frac{f}{1 - f} a_{ij} y_j \geq f\,.$$

Derive a GMIC for the mixed-integer set

$$S = \{(x, y) \in \mathbb{Z}^1_+ \times \mathbb{R}^2_+ \mid \frac{3}{2}x_1 + \frac{1}{7}y_1 + \frac{2}{7}y_2 = \frac{20}{7}\}$$

Solution: We have $f = 6/7$ and $f_{x_1} = 1/2$. The resulting GMIC is

$$\frac{1}{2}x_1 + \frac{1}{7}y_1 + \frac{2}{7}y_2 \geq \frac{6}{7}.$$

(c) (3 points) Indicate for each of the claims below whether they are true or false. No further motivation is needed.

- Branch-and-bound is a polynomial algorithm for integer optimization in fixed dimension.
- The basis in $\mathbb{R}^2$ consisting of the basis vectors $\boldsymbol{b}^1 = (0,\ 2)^T$ and $\boldsymbol{b}^2 = (1,\ 0)^T$ is reduced.
- The determinant of a lattice L depends only on L and not on the choice of basis used in the representation of L.

(Hint: Recall that a basis in $\mathbb{R}^2$ is reduced if

$$|\mu_{21}| = |\frac{(\boldsymbol{b}^2)^T \boldsymbol{b}^{1*}}{\|\boldsymbol{b}^{1*}\|^2}| \le \frac{1}{2}$$

and

$$\|\boldsymbol{b}^2 + \mu_{21}\boldsymbol{b}^{1*}\|^2 \ge \frac{3}{4}\|\boldsymbol{b}^{1*}\|^2 \,.)$$

Solution:

- False (see page 37 of Lecture4.pdf)
- False ($\mu_{21} = 0$ since the two vectors are orthogonal, but the second condition is not satisfied as $1 \not\ge \frac{3}{4} \cdot 4$.)
- True ($\det(L) := \sqrt{\det(\boldsymbol{B}^T\boldsymbol{B}}$, where $\boldsymbol{B}$ is *any* basis for L.)

2. (a) (4 points) et $N = \{1, \ldots, n\}$ be a set of items. The knapsack polytope is defined as $S_K := \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \le b\}$. Assume that all input is positive and integer. A set $C \subseteq N$ is called a *cover* if $\sum_{j \in C} a_j > b$. Given a cover C, the extension set $E(C)$ is defined as $E(C) = C \cup \{j \in N \setminus C \mid a_j \ge a_k$ for all $k \in C\}$. Prove that the *extended* cover inequality $\sum_{j \in E(C)} x_j \le |C| - 1$ is valid for S_K.

Solution: See page 3 of lecture4.pdf.

(b) (2 points) Given is a knapsack set

$$26x_1 + 23x_2 + 19x_3 + 15x_4 + 14x_5 + 11x_6 + 8x_7 \le 43\,.$$

Derive two extended cover inequalities for this set. For each of the inequalities, specify the cover C, the extension set $E(C)$, and the inequality.

Solution: Take for instance:

- $C = \{3,4,6\}$, $E(C) = \{1,2,3,4,6\}$ yielding the inequality $x_1+x_2+x_3+x_4+x_6 \le 2$,
- $C = \{4,5,6,7\}$, $E(C) = \{1,2,3,4,6,7\}$ yielding the inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \le 3$,

(c) (4 points) Let $N := \{1, \ldots, n\}$. Consider the following set:

$$S_K^{\geq} := \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \geq b\}\,.$$

Assume that

- $a_j \in \mathbb{Z}_+$ for all $j \in N$,
- $b \in \mathbb{Z}_+$,
- $\sum_{j \in N} a_j - a_k > b$ for all $k \in N$.

Determine the dimension of conv$(S_K^{\geq})$.

Solution: Recall that the dimension of a set S is equal to the affine dimension of S minus one. Notice that the origin belongs to the affine hull of conv$(S_K^{\geq})$. Hence, the affine dimension of conv$(S_K^{\geq})$ is equal to the linear dimension of conv$(S_K^{\geq})$ plus one. All the vectors

$$\begin{aligned} x_j &= 1, \ \forall j \in N \setminus \{k\} \\ x_k &= 0 \end{aligned}$$

for $k = 1, \ldots, n$ are feasible due to the assumptions, and they are linearly independent. There are $n = |N|$ of these vectors. Hence, the affine dimension of conv$(S_K^{\geq})$ is $\geq n+1$, and therefore dim conv$(S_K^{\geq}) \geq n$. Since $S_K \subset R^n$, we know that dim conv$(S_K^{\geq}) \leq n$. Combining the two yields that conv$(S_K^{\geq})$ is equal to n.

3. Consider the quadratic assignment problem $QAP(A, B)$:

$$z^* = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)},$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are real $n \times n$ matrices, and $\mathcal{S}_n$ denotes the set of all permutations of $\{1, \ldots, n\}$. Recall the following notation:

$$\langle v, u \rangle^- = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} v_i u_{\varphi(i)} \qquad v, u \in \mathbb{R}^n.$$

(a) (3 points) Given a graph $G = (V, E)$ with $|V| = n$, we would like to know if G is triangle-free, i.e. whether G contains K_3 as an induced subgraph or not.
Explain how you may answer this question by solving a quadratic assignment problem of the form $QAP(A, B)$, i.e. define suitable matrices A and B. Motivate your answer.

Solution: Define the matrix:

$$B = (b_{ij}) := \begin{bmatrix} 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

and A the adjacency matrix of G.
Consider $QAP(A, B)$:

$$z^* = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)},$$

Now G is triangle-free if and only if $z^* > -6$.

(b) (7 points) Consider problem $QAP(A, B)$ in the case where the diagonal entries of A and B are all zero. Denote the off-diagonal elements in row i of A by $\hat{a}_i$, e.g.

$$\hat{a}_1 = [a_{12}\ a_{13}\ \ldots\ a_{1n}].$$

Prove that

$$\min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{\varphi(i)\varphi(k)} \geq \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} (\langle \hat{a}_i, \hat{b}_{\varphi(i)} \rangle^-,$$

i.e. that the Gilmore-Lawler bound is a lower bound for $QAP(A, B)$.

Solution: The derivation is given in §7.5.1 in the book *Assignment Problems*. In the slides: Slide 13 of Lecture 7.

4. (a) (4 points) Consider a given instance of $QAP(A, B)$ where there are vectors $u, v \in \mathbb{R}^n$ such that the matrix $A = (a_{ij})$ is given by

$$a_{ij} = u_i + v_j \quad \forall i, j \in \{1, \ldots, n\}.$$

Show that problem $QAP(A, B)$ reduces to a linear sum assignment problem, i.e. one may obtain the optimal value as $z^* = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} c_{i\varphi(i)}$ for some suitable matrix $C = (c_{ij})$. Give the expression for the matrix C in terms of u, v and B, and motivate your answer.

Solution: Proof of Proposition 8.2 in the book. The expression for $C = (c_{ij})$:

$$c_{ij} = u_i \sum_{\ell=1}^{n} b_{j\ell} + v_i \sum_{\ell=1}^{n} b_{\ell j}.$$

(b) (3 points) Solve the following instance of $QAP(A, B)$ using any method of your choice:

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 1 & 0 & 5 \end{pmatrix}$$

Solution: Note that $a_{ij} = v_i + v_j, v = [1\ 2\ 3\ 4]'$. Now use Proposition 8.2.
One gets an equivalent LSAP with

$$C = 2 \times \begin{pmatrix} 7 & 7 & 7 & 7 \\ 14 & 14 & 14 & 14 \\ 21 & 21 & 21 & 21 \\ 28 & 28 & 28 & 28 \end{pmatrix}$$

Thus all permutations are optimal, and $z^* = 140$.

(c) (3 points) Consider $QAP(A, B)$, and set

$$z(\varphi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)}.$$

Recall that

$$\mathsf{Aut}(A) = \left\{ \varphi \in \mathcal{S}_n \ : \ a_{ij} = a_{\varphi(i)\varphi(j)} \ \forall i, j \right\}.$$

Prove that, if $\varphi \in \mathsf{Aut}(A) \cup \mathsf{Aut}(B)$, then $z(\varphi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}$.

Solution: The required result in the case $\varphi \in \mathsf{Aut}(B)$ is an immediate consequence of the definition of $\mathsf{Aut}(B)$.
So assume $\varphi \in \mathsf{Aut}(A)$. Now $\varphi^{-1} \in \mathsf{Aut}(A)$ since $\mathsf{Aut}(A)$ is a group.
Note that

$$\begin{aligned} z(\varphi) &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)} \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} a_{\varphi^{-1}(k)\varphi^{-1}(l)} b_{kl} \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} b_{kl}, \end{aligned}$$

as required.

5. In *protein interaction networks*, each vertex represents a protein while edges represent interaction between certain proteins. When analysing such networks, one is generally interested in finding groups of proteins with a lot of interaction. Formally, such groups are called n-*clubs* and defined as induced subgraphs with diameter at most n (where the *diameter* of a (sub)graph is the maximum distance between any pair of nodes). For the special case of $n = 2$, the associated problem is formulated as follows:

2-CLUB
Instance: graph $G = (V, E)$ and number $k \in \mathbb{N}$.
Parameter: k

Question: is there a set $S \subseteq V$ of at least k vertices such that the subgraph of G induced by S has diameter at most 2?

(a) (5 points) Consider the following reduction rule for the 2-CLUB problem. If there is a vertex v with degree at least $k-1$, then delete all vertices except for v and its neighbours.

Prove that this reduction rule is safe, i.e., the reduced instance is a yes-instance if and only if the original instance is a yes-instance.

> **Solution:** First suppose that there does exist a vertex v of degree at least $k-1$. Then the original instance is a yes-instance because the subgraph induced by v and its neighbours has diameter at most 2. The reduced instance is also a yes-instance for the same reason. If, on the other hand, there does not exist a vertex v of degree at least $k-1$, then the instance is not modified and hence the claim is trivially true.

(b) (5 points) Show that, if the reduction rule above cannot be applied, one can solve the instance by solving $|V|$ smaller instances with $O(k^2)$ vertices each. (Such an approach is called *Turing kernelization*.)

> **Solution:** Consider each vertex v in turn. If there exists a 2-club containing v, then this 2-club can only contain vertices with distance at most 2 from v. Since the reduction rule is not applicable, each vertex has degree at most $k-2$. Hence there are at most $1+(k-2)+(k-2)(k-3)$ vertices with distance at most 2 from v. The subgraph induced by these vertices is a smaller instance of size $O(k^2)$. Moreover, the original instance is a yes-instance if and only if at least one of the smaller instances is a yes-instance.

6. Consider the following problem.

DOMINATING SET
Instance: graph $G = (V, E)$ and number $k \in \mathbb{N}$.
Parameter: k
Question: Is there a set $S \subseteq V$ of at most k vertices such that each vertex not in S is adjacent to at least one vertex in S?

(a) (5 points) Show that, for each constant Δ, the DOMINATING SET problem is FPT for graphs with maximum degree Δ. Also analyse the running time of your algorithm.

> **Solution:** Consider an arbitrary vertex v. If v is not in S, then at least one of its neighbours needs to be in S. Hence, we can split into at most $\Delta+1$ subproblems, in each subproblem either v or one of its neighbours is deleted and k is reduced by one, and recurse. If there are no vertices left then the problem is solved trivially. The overall running time is $O((\Delta+1)^k(|V|+|E|))$.

(b) (5 points) Show that the DOMINATING SET problem is FPT for planar graphs. Again analyse the running time of your algorithm.

Solution: Euler's formula states that $f + n = m + 2$ with f the number of facets, n the number of vertices and m the number of edges. Since any edge is in at most two facets and each facet is surrounded by at least three edges, we have $3f \leq 2m$. Hence,

$$m = n + f - 2 \leq n + \frac{2}{3}m - 2$$

and hence

$$m \leq 3n - 6$$

The sum of the degrees is $2m$, so the average degree is

$$\frac{2m}{n} \leq \frac{6n - 12}{n} < 6.$$

Since the average degree is less than 6, there must be a vertex with degree at most 5.

Consider a vertex v of degree at most 5. If v is not in S, then at least one of its neighbours needs to be in S. Hence, we can split into at most 6 subproblems, in each subproblem either v or one of its neighbours is deleted and k is reduced by one, and recurse. If there are no vertices left then the problem is solved trivially. The overall running time is $O(6^k(|V| + |E|))$.

End of test