Exam WI4410 Advanced Discrete Optimization

June 26, 2019, 13:30–16:30

The exam consists of 6 questions. In total you can obtain 60 points. Your grade is calculated by dividing the number of points you obtained by 6. You may use a non-graphical calculator during the exam. Using a graphical calculator, notes, phone, smart-watch, etc. is **not** permitted. The total number of pages of this exam is 7.

Please, write your answers to questions 1-2, 3-4, and 5-6 on separate sheets of paper. Good luck!

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1. (a) (3 points) Given is a set $N = \{1, \ldots, n\}$. Consider the following knapsack set.

$$X_K = \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \leq b\}.$$

Assume that $a_j < b$ for all $j \in N$, and that $a_j \in \mathbb{Z},\ \forall j \in N, b \in \mathbb{Z}$. Determine dim(conv X_K).

Solution: Due to the assumptions, all n-dimensional unit vectors belong to conv X_K, and they are linearly independent. The n unit vectors together with the zero-vector (that also belongs to conv X_K) are affinely independent. We therefore have $n+1$ affinely independent feasible vectors, and hence dim(conv X_K) $= n$. (Here I have also given full points to students who assume $a_j \in \mathbb{Z}_+$ for all $j \in N$. This was actually the intention and is reflected in my solution here.)

(b) (3 points) Consider the knapsack cover inequalities for X_K,

$$\sum_{j \in C} x_j \leq |C| - 1,$$

where $C \subseteq N$ is a subset such that $\sum_{j \in C} a_j > b$. The set $C \subseteq N$ is called a *cover*. Prove that these cover inequalities are valid for conv X_K.

Solution: Let $\boldsymbol{x}^R$ denote a $\{0,1\}^n$-vector with $x_j^R = 1$ if $j \in R$ and $x_j^R = 0$ otherwise. Suppose $\boldsymbol{x}^R \in X_K$ and that $\boldsymbol{x}^R$ is such that $\sum_{j \in C} x_j^R \geq |C|$. This implies that $R \cap C = C$. Therefore, $\sum_{j \in R} a_j \geq \sum_{j \in R \cap C} a_j = \sum_{j \in C} a_j > b$, where the inequality is due to the definition of a cover. The fact that $\sum_{j \in R} a_j > b$ contradicts that $\boldsymbol{x}^R$ is feasible.

(c) (2 points) Consider the following knapsack set:

$$X_K^1 = \{\boldsymbol{x} \in \{0,1\}^7 \mid 7x_1 + 9x_2 + 14x_3 + 5x_4 + 11x_5 + 17x_6 + 4x_7 \leq 19\}.$$

Derive two knapsack cover inequalities for conv X_K^1.

> **Solution:** Take for instance $C = \{2,5\}$ with $a_2 + a_5 = 9 + 11 = 20 > 19$ and $C = \{1,2,7\}$ with $a_1 + a_2 + a_7 = 7 + 9 + 4 = 20 > 19$. The corresponding cover inequalities are:
>
> $$\begin{aligned} x_2 + x_5 &\leq 1 \\ x_1 + x_2 + x_7 &\leq 2. \end{aligned}$$

(d) (2 points) Given a cover $C \subseteq N$, the *extension set* $E(C)$ is defined as $E(C) := C \cup \{k \in N \setminus C \mid a_k \geq a_j,\ \forall j \in C\}$. The *extended cover inequalities* for knapsack sets are:

$$\sum_{j \in E(C)} x_j \leq |C| - 1.$$

Derive one extended cover inequality for conv X_K^1, with X_K^1 given in (c).

> **Solution:** Take for instance $C = \{2,5\}$. This yields $E(C) = C \cup \{3,6\}$ and the extended cover inequality
>
> $$x_2 + x_3 + x_5 + x_6 \leq 1.$$

(e) (3 points) Given is a knapsack set

$$X_K = \{\boldsymbol{x} \in \{0,1\}^6 \mid 35x_1 + 25x_2 + 15x_3 + 20x_4 + 15x_5 + 10x_6 \leq 65\}.$$

The knapsack cover inequality $x_2 + x_4 \leq 1$ is valid for the set

$$\text{conv } X_K \cap \{x \in \mathbb{R}^6 \mid x_1 = 1,\ x_3 = x_5 = x_6 = 0\}).$$

Apply maximal lifting to the variable x_5.

> **Solution:**
>
> $$\alpha x_5 + x_2 + x_4 \leq 1,$$
>
> $$\alpha \leq 1 - \max\{x_2 + x_4 \mid 25x_2 + 20x_4 \leq 65 - 35 - 15 = 15\},$$
>
> $$\alpha \leq 1 - 0 = 1.$$
>
> Choose maximal value of α, i.e., $\alpha = 1$. This yields the inequality
>
> $$x_2 + x_4 + x_5 \leq 1.$$

2. (a) (4 points) Consider the following pure integer linear set:

$$S = \{\boldsymbol{x} \in \mathbb{Z}_+^2 \mid -x_1 + x_2 \leq 0,\ x_1 + x_2 \leq 3\}.$$

Derive graphically a split inequality for this set, i.e., given the figure that you draw, give the split disjunction and motivate why the derived inequality belongs to the family of split inequalities.

Solution: Take for instance the split disjunction $x_1 \leq 1$ or $x_1 \geq 2$. Let P be the linear relaxation of S. The inequality $x_2 \leq 1$ is valid for $P \cap \{\boldsymbol{x} \mid x_1 \leq 1\}$ and for $P \cap \{\boldsymbol{x} \mid x_1 \geq 2\}$.

(b) (3 points) Indicate whether the following statements are "true" or "false". No motivation is needed.

(i) Suppose we are given a single-row pure integer set $S = \{\boldsymbol{x} \in \mathbb{Z}_+^n \mid \sum_{j=1}^n a_j x_j = b\}$ with $b \notin \mathbb{Z}$. The Gomory mixed-integer inequality (GMIC) derived for this set is at least as strong as the Gomory fractional cut derived from the same set.

(ii) The basis vectors produced by the LLL lattice basis reduction algorithm are nearly orthogonal.

(iii) Lenstra's algorithm for integer programming is polynomial, also for varying number of variables.

Solution: (i): True, (ii): True, (iii): False

3. Consider the quadratic assignment problem $QAP(A, B)$:

$$z^* = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)},$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are real $n \times n$ matrices, and $\mathcal{S}_n$ denotes the set of all permutations of $\{1, \ldots, n\}$.

(a) (3 points) A graph $G = (V, E)$ with $|V| = n$ is called *Hamiltonian* if it contains a cycle of length n. Explain how one may decide whether a given graph is Hamiltonian by solving the quadratic assignment problem $QAP(A, B)$ for suitable choices of the matrices A and B.

Solution: Define the matrix B as the adjacency matrix of C_n (the n-cycle), and and A as the adjacency matrix of G.
Consider $QAP(A, B)$:

$$z^* = \max_{\varphi \in \mathcal{S}_n} \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)},$$

with optimal permutation φ^*. Now G is Hamiltonian iff $z^* = n$.

(b) (3 points) Consider problem $QAP(A, B)$ in the case where $A = aa^T$ and $B = bb^T$ for some vectors $a, b \in \mathbb{R}^n$ with $n \geq 2$. Show that the eigenvalue bound for problem $QAP(A, B)$ is always zero in this case, but that the optimal value z^* can be arbitrarily large.

Solution: The matrix A only has one nonzero eigenvalue given by $\|a\|^2$. Similarly the matrix B only has one nonzero eigenvalue $\|b\|^2$.
Thus the eigenvalue bound is given by

$$0 \cdot \|b\|^2 + 0 \cdot 0 + \ldots + 0 \cdot 0 + \|a\|^2 \cdot 0 = 0.$$

By letting J denote the all-ones matrix, and $A = kJ$ and $B = J$ for some $k > 0$, then $z^* = kn^2$, that can be arbitrarily large, since $k > 0$ is arbitrary.

(c) (4 points) Let

$$A = \begin{pmatrix} 0 & 4 & 2 \\ 8 & 0 & 6 \\ 12 & 10 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 3 & 6 \\ 3 & 0 & 3 \\ 6 & 3 & 0 \end{pmatrix},$$

and calculate the Gilmore-Lawler lower bound for the resulting instance $QAP(A, B)$. (You may solve the linear assignment problem by inspection.) Also state whether the Gilmore-Lawler lower bound equals z^* here, and motivate your answer.

Solution: The Gilmore-Lawler lower bound equals 150 for this instance, and is obtained from the linear assignment problem with matrix:

$$6 \cdot \begin{pmatrix} 4 & 3 & 4 \\ 10 & 7 & 10 \\ 16 & 11 & 16 \end{pmatrix}$$

with optimal permutation $\varphi = (3\ 1\ 2)$ or $\varphi = (1\ 3\ 2)$. The optimal value is $z^* = 162$ corresponding to the same permutations. Thus the Gilmore-Lawler lower bound does not equal z^* here.

4. In this question we again consider $QAP(A, B)$ as defined in the previous question. Recall that each node of the *polytomic branching tree* corresponds to a partial assignment of facilities to locations, and its child nodes are created by assigning one unassigned facility to each available location in turn.

(a) (5 points) Show that the number of nodes in the polytomic branch-and-bound tree equals $\sum_{k=0}^{n} \binom{n}{k} k!$ and the number of leaves equals $n!$. Also show #nodes/#leaves $\leq e \approx 2.71828$, with equality in the limit as $n \to \infty$.

Solution:

At level $k = 0$ there is one root node: $1 = \binom{n}{0} 0!$.

At level $k = 1$ there n children of the root node: $n = \binom{n}{1} 1!$.

Each of the n nodes at level 1 has $n - 1$ children.

So at level $k = 2$ there $n(n-1)$ nodes: $n(n-1) = \binom{n}{2} 2!$.

Finish the proof by induction: assuming that at level k there are $\binom{n}{k}k!$ nodes, we show that at level $k+1$ there are $\binom{n}{k+1}(k+1)!$, since each node at level k has $n-k$ children.
Indeed,

$$\binom{n}{k}k! \times (n-k) = \frac{n!}{(n-k)!}(n-k) = \frac{n!}{(n-k-1)!} = \binom{n}{k+1}(k+1)!.$$

Thus the total number of nodes is $\sum_{k=0}^{n}\binom{n}{k}k! = \sum_{k=0}^{n}\frac{n!}{k!}$ (summing over all levels of the tree).
Each leaf corresponds to a unique $\varphi \in \mathcal{S}_n$, so there are $|\mathcal{S}_n| = n!$ leaves.
Thus

$$\#\text{nodes}/\#\text{leaves} = \left(\sum_{k=0}^{n}\frac{n!}{k!}\right)/n! = \sum_{k=0}^{n}\frac{1}{k!}$$

Recalling that

$$e = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}$$

completes the proof.

(b) (3 points) Assume $A = uu^T$ and $B = vv^T$ for nonnegative vectors $u, v \in \mathbb{R}^n_+$. Prove that $QAP(A, B)$ may be solved in polynomial time in this case.

Solution: Proposition 8.9 in the book.

(c) (2 points) Solve the following instance of $QAP(A, B)$ by the method of your choice, and give the optimal value z^* as well as the optimal permutation:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 16 & 12 & 4 \\ 12 & 9 & 3 \\ 4 & 3 & 1 \end{pmatrix}.$$

Solution: One has $A = uu^T$ and $B = vv^T$ with

$$u = [1\ 2\ 1]^T,\ v = [4\ 3\ 1]^T.$$

Thus (proof of Proposition 8.9):

$$z(\varphi^*) = (< u, v >^-)^2 = (1\cdot 4 + 1\cdot 3 + 2\cdot 1)^2 = 9^2 = 81,$$

corresponding to the optimal solution $\varphi^* = (1, 3, 2)$.

5. Consider the following problem.

ODDSUBGRAPH
Instance: graph $G = (V, E)$ and an odd integer k
Parameter: k

Question: does G have a subgraph $G' = (V', E')$ with $|E'| = k$ such that each $v \in V'$ has odd degree in G'?

Consider the following three reduction rules:

- (R1): if G has a matching of size k (a *matching* is a subset $M \subseteq E$ such that $e \cap f = \emptyset$ for all $e, f \in M$ with $e \neq f$), delete all vertices that are not incident to an edge of the matching;
- (R2): if G has a vertex v of degree at least k, delete all vertices except for v and k of its neighbours;
- (R3): if G has a vertex of degree 0, delete it.

(a) (5 points) Prove that each of these reduction rules is safe, i.e. show that a reduced instance is a yes-instance if and only if the original instance is a yes-instance.

Solution: (R1): if G has a matching of size k then the subgraph consisting of this matching has k edges and each vertex in the subgraph has degree 1, which is odd. So the original and the reduced instance are yes-instances.
(R2): if G has a vertex of degree at least k, the subgraph consisting of this vertex and k of its neighbours has k edges and each vertex in the subgraph has odd degree (1 or k). So, again, the original and reduced instance are yes-instances.
(R3): a vertex with degree 0 cannot be part of a subgraph in which each vertex has odd degree. So, if the original instance is a yes-instance, the same subgraph can be used to show that the reduced instance is a yes-instance and vice versa.

(b) (5 points) Prove that, if none of (R1), (R2) and (R3) is applicable, there are at most $2k^2$ vertices left. Hence, ODDSUBGRAPH has a quadratic kernel.

Solution: Take any maximal matching M (which can be found by adding edges to the matching until no edge can be added). Let U denote the set of vertices that are incident to an edge of M. Since (R1) is not applicable, $|M| \leq k$ and hence $|U| \leq 2k$. Each vertex has degree at most k since (R2) is not applicable. Each vertex is either in U or has a neighbour in U because otherwise we could either extend the matching (if there is a vertex not in U with a neighbour that is also not in U) or rule (R3) would be applicable (if there is a vertex with no neighbours). So, in total, there are at most $2k + 2k(k-1) = 2k^2$ vertices left.

6. Consider the following problem.

COLOURFULPATH
Instance: graph $G = (V, E)$, integer k and function $f : V \to \{1, \dots, k\}$ (where $1, \dots, k$ can be interpreted as colours)
Parameter: k
Question: does G have a path P such that for each $i \in \{1, \dots, k\}$ there is exactly one vertex v on P with $f(v) = i$ (i.e., a path using each colour exactly once)?

(a) (4 points) Let $C \subseteq \{1, \dots, k\}$ and $v \in V$. Define a C-*path* to be a path P containing $|C|$ vertices such that for each $i \in C$ there is exactly one vertex w on P with $f(w) = i$. Prove that there exists a C-path starting at v if and only if there exists a $C \setminus \{f(v)\}$-path starting at a neighbour of v.

Solution: First suppose there exists a C-path starting at v. Removing v from the path gives a $C \setminus \{f(v)\}$-path starting at a neighbour of v.
Now suppose there exists a $C \setminus \{f(v)\}$-path starting at a neighbour of v. Then this path doesn't use v since it only uses vertices with colours in $C \setminus \{f(v)\}$. So we can add v to this path and obtain a C-path starting at v.

(b) (6 points) Prove that ColourfulPath is FPT by describing a dynamic programming algorithm for it. Also determine the running time of your algorithm.

Solution: For $v \in V$ and $C \subseteq \{1, \dots, k\}$, define $g(C, v) = 1$ if there is a C-path starting in v and $g(C, v) = 0$ otherwise.
Initialization. For $|C| = 1$: $g(\{i\}, v) = 1$ if $f(v) = i$ and $g(\{i\}, v) = 0$ otherwise.
Recursion. For $|C| = 2, 3, \dots, k$:

$$g(C, v) = \max_{u \in N(v)} g(C \setminus \{f(v)\}, u),$$

with $N(v)$ the set of neighbours of v.
Solution. The answer is yes if and only if

$$\max_{v \in V} g(\{1, \dots, k\}, v) = 1$$

Correctness follows by part (a). For each vertex $v \in V$ and for each subset of $\{1, \dots, k\}$, we loop through all (at most $|V| - 1$) neighbours of v. So the running time is $O(2^k |V|^2)$.

End of test