Exam WI4410 Advanced Discrete Optimization

July 1, 2020, 09:00–12:00

The exam consists of 8 questions. In total you can obtain 60 points. Your grade is calculated by dividing the number of points you obtained by 6. This is an open-book exam. It is NOT allowed to discuss with anyone else. If you have any questions regarding the exam, or technical questions regarding uploading of your answer, please contact Karen Aardal, k.i.aardal@tudelft.nl. The total number of pages of this exam is 9. Good luck!

Please review the instructions posted on the announcement page for the course. The most important points are repeated below.

- Write your answers **by hand**.
- On your first answer sheet, you should write down the following statement:
  "This exam is solely undertaken by myself, without any assistance from others."
  Please **sign your name** below this statement. Next to this, when scanning your exam, you should **place your student ID**.
- Scan your work and submit it as **one single pdf-file** to Assignments on Brightspace.

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1. (a) (4 points) Consider the following integer optimization problem:

$$\begin{aligned} \max z = 5x_1 + 8x_2 & \\ \text{s.t.}\ \ x_1 + x_2 &\le 6 \\ 5x_1 + 9x_2 &\le 45 \\ \boldsymbol{x} &\in \mathbb{Z}^2_+ . \end{aligned}$$

Let s_1 and s_2 be integer slack variables in the above constraints. After solving the LP-relaxation of the problem we obtain:

$$\begin{array}{ccccccc} -z & & & -1.25s_1 & -0.75s_s & = & -41.25 \\ & x_1 & & +2.25s_1 & -0.25s_2 & = & 2.25 \\ & & x_2 & -1.25s_1 & +0.25s_2 & = & 3.75 \end{array}$$

Generate a Gomory fractional cut *and* a Gomory Mixed-Integer Cut from the second row of the Simplex tableau (the row in which x_1 is a basic variable).

Solution: $f_0 = 0.25,\ f_{s_1} = 0.25,\ f_{s_2} = -0.25 - (-1) = 0.75$. Gomory fractional cut:

$$0.25s_1 + 0.75s_2 \ge 0.25\,,$$

(or, equivalently:

$$s_1 + 3s_2 \ge 1\,.)$$

Gomory mixed-integer cut (GMIC): Notice here that $f_{s_2} > f_0$. This yields the cut

$$0.25s_1 + \frac{0.25 \cdot 0.25}{0.75}s_2 \ge 0.25\,.$$

(or, equivalently:

$$s_1 + \frac{1}{3}s_2 \ge 1\,.)$$

(b) (2 points) Which of the two inequalities in the previous exercise (1(a)) is stronger? A brief motivation suffices.

Solution: Since the right-hand side in both cuts are the same we look at the coefficients of the left-hand side. Here we have that the coefficient for s_1 is the same in both inequalities, but that the coefficient for s_2 is smaller in the GMIC than in the fractional cut, which implies that the GMIC is stronger. In general it holds that GMIC generated from a certain row is at least as strong as the Gomory fractional cut generated from the same row.

(c) (4 points) Consider the following integer optimization problem:

$$\begin{aligned} \max x_2 & \\ \text{s.t.}\ -x_1 + 2x_2 &\le 2 \\ 2x_1 + x_2 &\le 10 \\ x &\in \mathbb{Z}^2_+ \end{aligned}$$

The extreme points of the LP-relaxation are:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3\frac{3}{5} \\ 2\frac{4}{5} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Derive one split cut that cuts off the non-integer extreme point. Please, mention the split disjunction that you use *explicitly* in your answer. You may solve the question graphically, but you may also derive the inequality using e.g. Lemma 5.4 in the book. (See also slide 33 of Lecture 2.) In case you solve the question graphically, include the figure with explanations in your answer.

Solution: Here I give the derivation for the split disjunction $x_1 \leq 3 \vee x_1 \geq 4$, since that is a bit more involved. (Another possibility is to consider the split disjunction $x_2 \leq 2 \vee x_2 \geq 3$.) I use the same notation as in Lemma 5.4. Take $\pi = (1,\ 0)$ (since I split on variable x_1) and solve the system of equations

$$u^\mathsf{T} A := \pi\,.$$

That yields the vector $u^\mathsf{T} = (-1/5,\ 2/5)$, which we then can split in the vectors u^+ and u^- with $(u^+)^\mathsf{T} = (0,\ 2/5)$ and $(u^-)^\mathsf{T} = (1/5,\ 0)$. We also obtain $f = u^\mathsf{T} b - \lfloor u^\mathsf{T} b \rfloor = 3/5$. Then, just using inequality (5.6) just above Lemma 5.4 yields the split cut

$$x_1 + 2x_2 \leq 8\,.$$

The disjunction $x_2 \leq 2 \ \vee \ x_2 \geq 3$ yields the split cut

$$x_2 \leq 2\,.$$

2. (a) (4 points) Let $N := \{1, \dots, n\}$. Consider the following set:

$$S_K^{\geq} := \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \geq b\}.$$

Assume that

- $a_j \in \mathbb{Z}_+$ for all $j \in N$,
- $b \in \mathbb{Z}_+$,
- $\sum_{j \in N} a_j - a_k > b$ for all $k \in N$.

Determine the dimension of conv $S_K^{\geq}$.

Solution: Recall that the dimension of a set S is equal to the affine dimension of S minus one. Notice that the origin belongs to the affine hull of conv($S_K^{\geq}$). Hence, the affine dimension of conv($S_K^{\geq}$) is equal to the linear dimension of conv($S_K^{\geq}$) plus one. All the vectors

$$\begin{aligned} x_j &= 1, \ \forall j \in N \setminus \{k\} \\ x_k &= 0 \end{aligned}$$

for $k = 1, \dots, n$, are feasible due to the assumptions, and they are linearly independent. There are $n = |N|$ of these vectors. Hence, the affine dimension of conv($S_K^{\geq}$) is $\geq n+1$, and therefore dim conv($S_K^{\geq}$) $\geq n$. Since $S_K \subset R^n$, we know that dim conv($S_K^{\geq}$) $\leq n$. Combining the two yields that conv($S_K^{\geq}$) is equal to n.

(b) (3 points) Given is a ground set $N = \{1, \dots, n\}$ and a knapsack set $S_K = \{x \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \leq b\}$. For a given cover $C \subseteq N$, the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for S_K. Consider the instance $S_K = \{x \in \{0,1\}^4 \mid 19x_1 + 28x_2 + 12x_3 + 16x_4 \leq 39\}$. The cover inequality $x_1 + x_3 + x_4 \leq 2$ is valid for $S_K \cap \{x \mid x_2 = 0\}$. Apply maximal lifting to the variable x_2 and give the resulting lifted knapsack cover inequality.

Solution: Lift x_2:

$$\alpha x_2 + x_1 + x_3 + x_4 \leq 2,$$

i.e.,

$$\alpha \leq 2 - (x_1 + x_3 + x_4).$$

Applying maximal lifting yields

$$\alpha = 2 - \max\{x_1 + x_3 + x_4 \mid 19x_1 + 12x_3 + 16x_4 \leq 11\},$$

so, $\alpha = 2$. This gives the lifted inequality: $x_1 + 2x_2 + x_3 + x_4 \leq 2$.

(c) (3 points) Let $N = \{1, \ldots, n\}$. Consider the single-node flow set:

$$S_{SNF} := \{(\boldsymbol{x},\ \boldsymbol{y}) \in \mathbb{R}^n_+ \times \{0,1\}^n \mid \sum_{j \in N} x_j = b,\ x_j \leq u_j y_j\}.$$

A set $C \subseteq N$ is a *flow cover* if $\sum_{j \in C} u_j > b$. Let $\lambda := \sum_{j \in C} u_j - b$ and $(u_j - \lambda)^+ := \max(u_j - \lambda,\ 0)$. The flow cover inequality

$$\sum_{j \in C} x_j + \sum_{j \in C} (u_j - \lambda)^+ (1 - y_j) \leq b$$

is valid for conv S_{SNF}.

Suppose $u_j = u$ for all $j \in N$, and suppose that b is not a divisor of any multiple of u. What is the size of every cover set C that yields a non-trivial flow cover inequality, i.e., a flow cover inequality that is not part of, or dominated by, inequalities that define the set S_{SNF}. Motivate your answer.

Solution: The size of every cover set C that yields a non-trivial flow cover inequality is k, where k is the smallest integer such that $k \cdot u > b$,. i.e., $k = \lceil b/u \rceil$.

This yields $\lambda = k \cdot u - b$ and $(u - \lambda)^+ = u - k \cdot u + b = b - u(k-1) > 0$, where the last inequality holds since b is not a divisor of any multiple of u, and since k is the smallest integer such that $k \cdot u > b$. Since $(u - \lambda)^+ > 0$ the inequality is non-trivial.

A smaller size of a cover than k is not possible, as the resulting set would not be a cover. If we choose a larger set, say a set of size $l = k+1$, we would obtain $\lambda = l \cdot u - b = (k+1)u - b$, and $(u - \lambda) = u - (k+1)u + b = b - k \cdot u < 0$ as $k \cdot u > b$. The flow cover inequality would then reduce to $\sum_{j \in C} x_j \leq b$, since$(u - \lambda)^+ = 0$ for all $j \in C$. The inequality $\sum_{j \in C} x_j \leq b$ is dominated by the defining inequality $\sum_{j \in C} x_j = b$.

3. (5 points) Consider a linear system $Ax = b$, where A is a rational matrix of full row rank and b is a rational vector. Suppose we have an algorithm to compute the Hermite normal form of A as well as the unimodular matrix that transforms A into its Hermite normal form. How can we use this to find an integral solution x if it exists, or to establish no such solution exists?

Solution: Let $(B|0)$ be the Hermite normal form of A and let U be the unimodular matrix such that $(B|0) = AU$. Since B is lower triangular with nonzero diagonal the system $By = b$ has a unique solution that can be found by forwardsubstitution. Let

$$x = U \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

Then x is a solution to the original linear system:

$$Ax = AU \begin{pmatrix} y \\ 0 \end{pmatrix} = (B|0) \begin{pmatrix} y \\ 0 \end{pmatrix} = By = b.$$

If y is integral, then x is an integral solution to $Ax = b$.

If there exists an integral solition x to $Ax = b$, then with

$$\begin{pmatrix} u \\ v \end{pmatrix} = U^{-1}x$$

it follows that u is an integral solution to $Bu = b$. This shows that if y is not integral, then $Ax = b$ does not admit an integral solution.

4. (5 points) Show how Minkowski's convex body theorem can be used to show that a lattice $\Lambda \subseteq R^2$ has a nonzero lattice point x with $\|x\|_1 \leq \sqrt{2\det(\Lambda)}$. (Here $\|x\|_1 = |x_1| + |x_2|$ is the 1-norm of x.) Is there a lattice for which this bound is sharp? If so, give a lattice basis of such a lattice.

Solution: Let

$$K_r = \{x \in \mathbb{R}^2 : \|x\|_1 \leq r\}.$$

Since K_r is a convex body that is symmetric about the origin, Minkowski's convex body theorem asserts that K_r contains a nonzero lattice point from Λ whenever

$$\mathrm{vol}(K_r) \geq 2^2 \det(\Lambda).$$

We have $\mathrm{vol}(K_r) = 2r^2$, so K_r contains a lattice point whenever $r \geq \sqrt{2\det(\Lambda)}$. In other words, Λ contains a nonzero vector x with

$$\|x\|_1 \leq \sqrt{2\det(\Lambda)}.$$

The following lattice shows the bound is sharp:

$$\Lambda = \Lambda(B), \quad \text{with} \quad B = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}.$$

5. (5 points) Consider the following QR-decomposition:

$$A = \begin{pmatrix} 1 & -3.5 & 5 \\ 4 & -3 & -24 \\ 7 & -2.5 & 13 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 11 \\ 4 & -1 & -22 \\ 7 & 1 & 11 \end{pmatrix} \begin{pmatrix} 1 & -0.5 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = QR.$$

Find the matrix B obtained by performing the normalization step of the LLL algorithm to A.

Solution: After substracting the second column twice from the third column we get

$$\begin{pmatrix} 1 & -3.5 & 12 \\ 4 & -3 & -18 \\ 7 & -2.5 & 18 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 11 \\ 4 & -1 & -22 \\ 7 & 1 & 11 \end{pmatrix} \begin{pmatrix} 1 & -0.5 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Adding the first column to the second column and substracting the first column from the third columns gives

$$\begin{pmatrix} 1 & -2.5 & 11 \\ 4 & 1 & -22 \\ 7 & 4.5 & 11 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 11 \\ 4 & -1 & -22 \\ 7 & 1 & 11 \end{pmatrix} \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since all off-diagonal entries in

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are strictly larger than -0.5 and at most 0.5 we are done with the LLL-normalization step. The matrix B thus reads

$$B = \begin{pmatrix} 1 & -2.5 & 11 \\ 4 & 1 & -22 \\ 7 & 4.5 & 11 \end{pmatrix}.$$

6. (5 points) Let P be a polytope of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ containing an integer point. How can Lenstra's algorithm be used to find such an integer point in polynomial time?

Solution: Use linear programming to find

$$\alpha_i = \min_{x \in P} x_i$$

and

$$\beta_i = \max_{x \in P} x_i.$$

Since linear programming is in NP, the binary encoding length of α_i and β_i will be polynomial in n and in the binary encoding length of A and b. For $i = 1, \ldots, n$ we can now use binary search and Lenstra's algorithm to find a value γ_i such that

$$P \cap \{x \in \mathbb{R}^n : x_k = \gamma_k \text{ for } k = 1, \ldots, i\}$$

contains an integer point. Since the binary encoding lengths of α_i and β_i are polynomial, the length $\beta_i - \alpha_i$ is at most exponential, hence binary search will give γ_i in polynomially many steps.

7. (8 points) Consider the following parameterized problem discussed in the course.

Covering Points by Lines
Instance: n points in the plane and a number $k \in \mathbb{N}$.
Parameter: k.
Question: do there exist k straight lines that cover all the points?

Prove that COVERING POINTS BY LINES is FPT by designing a **bounded-search tree** algorithm for it. Also analyse the running time of your algorithm.

Hint: observe that the problem is equivalent to partitioning the set of n points into k sets such that the points in each set lie on a straight line. You do not need to optimize any polynomial factors in the running time.

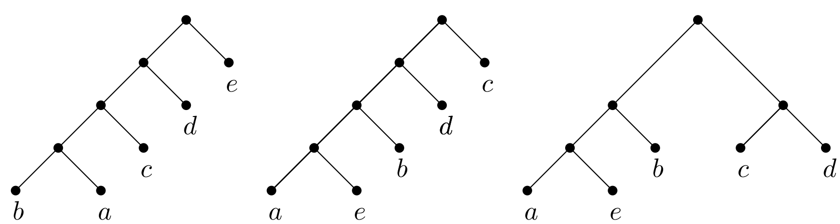
Solution: Initialise $L_1, \dots, L_k = \emptyset$. Pick an arbitrary point p that is not in any of $L_1, \dots, L_k$. For each $i = 1, \dots, k$, create a subproblem where p is put into L_i. If L_i now contains two points, we modify the subproblem by removing all points on the line through these two points, removing L_i and reducing k by one.

To analyse the running time, note that we branch into at most k subproblems. Moreover, the search tree is at most $2k$ deep since a set is removed once it contains at least two points. Hence, the total running time is $O(k^{2k}n)$.

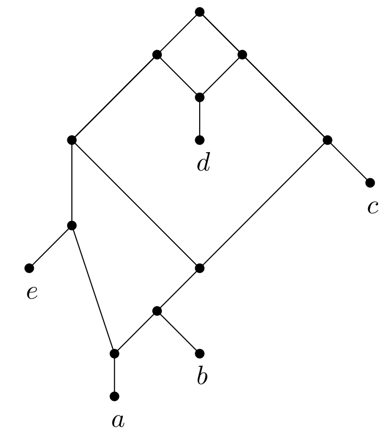
8. Consider the following parameterized problem discussed in the course.

TEMPORAL HYBRIDIZATION NUMBER
Given: a collection $\mathcal{T}$ of rooted binary phylogenetic trees and a natural number k
Parameter: k
Question: does there exist a temporal phylogenetic network that displays each tree from $\mathcal{T}$ and has reticulation number at most k?

(a) (6 points) Find a temporal phylogenetic network with reticulation number 3 displaying the rooted phylogenetic trees below.



Solution: The only cherry picking sequences of the trees are (a, b, d, c, e) and (a, b, d, e, c). Using either of those, and following the proof of the theorem of Humphries, Linz & Semple, you get the following network



(b) (6 points) Consider the following reduction rule. If each tree in $\mathcal{T}$ has S as a pendant subtree, then replace S by a single leaf x_S in each tree. The parameter k is unchanged. Prove that this rule is safe, i.e., that an original instance is a yes-instance if and only if the reduced instance is a yes-instance.

Solution: We use the characterization of the problem in terms of cherry picking sequences described in the theorem by Humphries, Linz & Semple.

First suppose the original instance is a yes-instance. Then there exists a weight-k cherry picking sequence s. All the leaves of S are picked with weight-0, except possibly the leaf x_ℓ of S that is picked last. Let s' be the cherry picking sequence obtained from s by removing all leaves of S except x_ℓ and replacing x_ℓ by x_S. This gives a weight-k cherry picking sequence for the reduced trees.

Now suppose the reduced instance is a yes-instance and let s be a weight-k cherry picking sequence. Let s' be the sequence obtained from s by replacing x_S by a cherry picking sequence of S. As before, all leaves of S are picked with weight-0, except possibly the leaf x_ℓ of S that is picked last. Hence, s' is a weight-k cherry picking sequence for the original trees.

End of test