Exam WI4410 Advanced Discrete Optimization

June 27, 2018, 13:30–16:30

The exam consists of 7 questions. In total you can obtain 60 points. Your grade is calculated by dividing the number of points you obtained by 6. You may use a non-graphical calculator during the exam. Using a graphical calculator, notes, phone, smart-watch, etc. is **not** permitted. The total number of pages of this exam is 6. Good luck!

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1. (a) (3 points) Given is the following set of points:

$$X = \{(0,0),\ (1,0),\ (2,0),\ (0,1),\ (1,1)\}$$

Give two valid formulations P_1 and P_2 for this set, with the property that $P_1 \subset P_2$.

Solution: This is one possibility: $P_1 = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 2,\ x_2 \leq 1\}$, $P_2 = \{x \in \mathbb{R}^2_+ \mid 5x_1 + 4x_2 \leq 10,\ x_2 \leq 1\}$. The inequality $x_1 + x_2 \leq 2$ strictly dominates $5x_1 + 4x_2 \leq 10$.

(b) (3 points) Show that dim(conv(X)) = 2 by exhibiting 3 affinely independent points in X. Do not forget to argue why the suggested points are affinely independent.

Solution: The points $(0,0),\ (1,0),\ (0,1)$ are affinely independent. The unit vectors $(1,0),\ (0,1)$ are clearly linearly independent, and adding the origin yields 3 affinely independent points.

(c) (2 points) The following exercise may be solved graphically. For the set X given in (a), give:

- a valid linear inequality that defines a 0-dimensional face of conv(X),
- a valid linear inequality that defines a facet of conv(X).

Solution: The inequality $x_1 \leq 2$ defines the 0-dimensional face $(2,0)$, and the inequality $x_1 + x_2 \leq 2$ defines a factet (1-dimensional face).

2. (a) (3 points) et $N = \{1, \ldots, n\}$ be a set of items. The knapsack polytope is defined as $S_K := \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \leq b\}$. Assume that all input is positive and integer. A set $C \subseteq N$ is called a *cover* if $\sum_{j \in C} a_j > b$. Given is the set

$$S_K = \{x \in \{0,1\}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}.$$

The inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is a facet-defining cover inequality for

$$S_K \cap \{x \in \{0,1\}^7 \mid x_1 = x_2 = x_7 = 0\}.$$

Apply maximal lifting to the variable x_1, and write down the inequality that is obtained after lifting has been applied.

Solution: Find maximal value of α such that the inequality $\alpha x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$ is valid when x_1 is not fixed to zero anylonger.

$$\alpha = 3 - \max\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} = 3 - 1 = 2\,.$$

After lifting, the inequality becomes: $2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$.

(b) (3 points) Consider again the knapsack polytope S_K. Given a cover C, the extension set $E(C)$ is defined as $E(C) = C \cup \{j \in N \setminus C \mid a_j \geq a_k \text{ for all } k \in C\}$. Prove that the *extended* cover inequality $\sum_{j \in E(C)} x_j \leq |C| - 1$ is valid for S_K.

Solution: See Exercise 1 from Lecture 3.

(c) (2 points) Gomory's mixed-integer cut (GMIC) for the mixed-integer set

$$S = \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p \mid \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b\}$$

is as follows:

$$\sum_{f_j \leq f} f_j x_j + \sum_{f_j > f} \frac{f(1-f_j)}{1-f} x_j + \sum_{g_j \geq 0} g_j y_j - \sum_{g_j < 0} \frac{f}{1-f} g_j y_j \geq f\,.$$

Derive a GMIC for the mixed-integer set

$$S = \{(x,y) \in \mathbb{Z}_+^1 \times \mathbb{R}_+^2 \mid x_1 + \frac{1}{7}y_1 - \frac{2}{7}y_2 = \frac{20}{7}\}$$

Solution: $f = \frac{6}{7}$, $g_1 = \frac{1}{7}$, $g_2 = -\frac{2}{7}$, $\frac{f}{1-f} = \frac{6/7}{1/7} = 6$. The GMIC for this row is:

$$\frac{1}{7}y_1 - 6(-\frac{2}{7})y_2 \geq \frac{6}{7} \quad \Rightarrow \quad \frac{1}{7}y_1 + \frac{12}{7}y_2 \geq \frac{6}{7} \quad \Rightarrow \quad y_1 + 12y_2 \geq 6\,.$$

(d) (2 points) Given is the mixed-integer optimization problem:

$$\begin{aligned} \max z = x + 2y & \\ \text{s.t.} -x + y &\leq 2 \\ x + y &\leq 5 \\ 2x - y &\leq 4 \\ y \geq 0,\ x &\in \mathbb{Z}_+ \end{aligned}$$

Derive, graphically, a split inequality for this problem.

> **Solution:** Take for instance the split disjunction $x \leq 1 \vee x \geq 2$. The inequality $y \leq 3$ is valid for the union of the two resulting polytopes and is therefore a split inequality.

3. (2 points) Indicate whether the following statements are true or false:

   - We can determine whether the rational system of inequalities $Ax \leq b$ has an integer solution in polynomial time if the dimension is fixed.
   - Suppose we are given a single-row pure integer set $S = \{\boldsymbol{x} \in \mathbb{Z}^n_+ \mid \sum_{j=1}^n a_j x_j = b\}$ with $b \notin \mathbb{Z}$. The Gomory fractional cut derived for this set is at least as strong as the Gomory mixed-integer cut (GMIC) derived from the same set.

> **Solution:** True, False

4. Consider the quadratic assignment problem $QAP(A, B)$:

$$z^* = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)},$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are real $n \times n$ matrices, and $\mathcal{S}_n$ denotes the set of all permutations of $\{1, \ldots, n\}$.

(a) (3 points) Given a graph $G = (V, E)$ with $|V| = n$ and an integer $k < n$, the densest k-subgraph problem is to find the densest induced subgraph in G with exactly k vertices. (I.e. to find the induced subgraph with k vertices having the maximum number of edges.)

Explain how you may obtain a densest k-subgraph of G by solving a quadratic assignment problem of the form $QAP(A, B)$, i.e. define suitable matrices A and B. Motivate your answer. In particular, explain how the densest k-subgraph may be constructed from the solution of $QAP(A, B)$.

> **Solution:** Define the matrix:
>
> $$B = (b_{ij}) := \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$
>
> where the upper left $k \times k$ principal submatrix is the $k \times k$ all-ones matrix, and A the adjacency matrix of G.
>
> Consider $QAP(A, B)$:
>
> $$z^* = \max_{\varphi \in \mathcal{S}_n} \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)},$$

with optimal permutation φ^*. Now the densest k-subgraph is induced by the vertices with labels $\varphi^*(1), \ldots, \varphi^*(k)$.

(b) (3 points) Consider problem $QAP(A, B)$ in the case where A and B are diagonal matrices. Show that the optimal value of $QAP(A, B)$ coincides with the eigenvalue bound in this case.

Solution: Assume the diagonal entries of A and B are a_{ii} and b_{ii} respectively. Then these are also the respective eigenvalues of A and B, so that the eigenvalue bound becomes

$$\min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} a_{ii} b_{\varphi(i)\varphi(i)}.$$

On the other hand, $QAP(A, B)$ reduces to

$$\begin{aligned} z^* &= \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)}, \\ &= \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} a_{ii} b_{\varphi(i)\varphi(i)}, \end{aligned}$$

which is the same expression as the eigenvalue bound.

(c) (4 points) Let

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 4 & 0 & 3 \\ 6 & 5 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix},$$

and calculate the Gilmore-Lawler lower bound for the resulting instance $QAP(A, B)$. (You may solve the linear assignment problem by inspection.) Also state whether the Gilmore-Lawler lower bound equals z^* here, and motivate your answer.

Solution: The Gilmore-Lawler lower bound equals 25 for this instance, and is obtained from the linear assignment problem with matrix:

$$\begin{pmatrix} 4 & 3 & 4 \\ 10 & 7 & 10 \\ 16 & 11 & 16 \end{pmatrix}$$

with optimal permutaion $\varphi = (3\ 1\ 2)$ or $\varphi = (1\ 3\ 2)$. The optimal value is $z^* = 27$ corresponding to the same permutations.

5. In this question we again consider $QAP(A, B)$ as defined in the previous question.

(a) (3 points) Show that:

$$z^* = \min_{X \in \mathbf{X}_n} \operatorname{tr}(AXB^T X^T),$$

where $\mathbf{X}_n$ is the set of $n \times n$ permutation matrices.

Solution: Solution sketched on Slide 19 of 1st lecture.

(b) (3 points) Use your result in the previous exercise to show that $QAP(A, B)$ may be solved in polynomial time if $A = A^T$ (symmetric) and $B = -B^T$ (skew-symmetrix).

Solution: In this case $z(\varphi) = 0$ for all permutations $\varphi \in \mathcal{S}_n$.

(c) (4 points) Show that

$$z^* = \text{tr}(S) + \text{tr}(T) + \min_{X \in \mathbf{X}_n} \left\{ x^T \left(B \otimes A - I \otimes S - T \otimes I \right) x \mid x = \text{vec}(X) \right\},$$

where T and S are any fixed, symmetric matrices. (**Hint**: first show that $\text{tr}(AXB^TX^T) = \text{vec}(X)^T(B \otimes A)\text{vec}(X)$).

Solution: See Slide 7 of Lecture 3, and Exercise 1 of Week 3.

6. Consider the following variant of the Closest String problem (where $d_h(s, s')$ is the number of positions where sequence s and sequence s' differ).

Instance: sequences $s_1, \ldots, s_k$ of length m over an alphabet Σ and a number $d \in \mathbb{N}$, with $d_h(s_1, s_2) = 2d$.
Parameter: d
Question: does there exist a sequence s of length m such that $d_h(s, s_i) \leq d$ for all $1 \leq i \leq k$?

(a) (2 points) Show that for each solution sequence s and for each position p holds that $s(p) = s_1(p)$ or $s(p) = s_2(p)$ (or both).

Solution: Suppose $s(p) \neq s_1(p)$ and $s(p) \neq s_2(p)$. Then, since $d_h(s, s_1) \leq d$, there are at most $d-1$ positions other than p where s and s_1 differ and, similarly, at most $d-1$ positions other than p where s and s_2 differ. Hence, there are at most $1+(d-1)+(d-1) = 2d-1$ positions where s differs from at least one of s_1, s_2. However, there are $2d$ positions where s_1 and s_2 differ and so s must differ from at least one of s_1, s_2 in each of these $2d$ positions, a contradiction.

(b) (3 points) Show that the problem can be solved in $O(m + kd4^d)$ time.

Solution: For each position p with $s_1(p) = s_2(p)$ we know that $s = s_1(p)$ by part (a). There are $2d$ positions p with $s_1(p) \neq s_2(p)$ and for each such position we know that either $s(p) = s_1(p)$ or $s(p) = s_2(p)$. Simply trying each possibility and checking whether it is a valid solution gives us a running time of $O(m + kd4^d)$.

7. A *tournament* is a directed graph $D = (V, A)$ with for each pair of vertices $u, v \in V$ exactly one of the arcs (u, v) and (v, u) in A. A *triangle* in D is a set of three arcs forming a directed cycle: $\{(u, v), (v, w), (w, u)\}$. Consider the following parameterized problem.

Arc Flipping in Tournaments
Instance: tournament $D = (V, A)$ and number $k \in \mathbb{N}$.

Parameters: k
Question: is it possible to reverse at most k arcs from D such that the resulting tournament is acyclic (i.e. has no directed cycle)?

(a) (5 points) Consider the following reduction rule. If an arc a is contained in at least $k+1$ triangles, then reverse a and reduce k by 1. Show correctness of this reduction rule, i.e. show that the obtained instance is a yes-instance if and only if the original instance is a yes-instance.

> **Solution:** First suppose the original instance is a yes-instance. If arc a is not reversed, then an arc other than a needs to be reversed in each of the at least $k+1$ triangles that a is in. Since these triangles only overlap in a, at least $k+1$ arcs would need to be reversed, a contradiction. Hence, arc a is reversed and at most $k-1$ other arcs. So the reduced instance is a yes-instance.
>
> Conversely, if the reduced instance is a yes-instance, then the original instance is also a yes-instance because we can simply reverse a and all arcs that are reversed in the solution to the reduced instance.

(b) (5 points) Consider the following reduction rule. If there is a vertex v that is not incident to any arc in a triangle, then delete v. Show correctness of this reduction rule.

> **Solution:** Clearly, if the original instance is a yes-instance then the reduced instance is a yes-instance as well because we can simply take the same solution.
>
> Now assume that the reduced instance is a yes-instance and suppose that reversing the same arcs in the original instance leaves us with a directed cycle. Then this directed cycle C passes through v. Note that $V \setminus \{v\}$ can be partitioned into two sets: the set Y of vertices with an arc to v and the set X of vertices with an arc from v. Since v is not in any triangle, all arcs between X and Y are directed from Y to X. Directed cycle C contains at least one of these arcs and so this arc has been reversed. However, if we instead choose not to reverse any of the arcs between Y and X the resulting directed graph is still acyclic. Hence, the original instance is also a yes-instance (we take the solution of the reduced instance but choose not to reverse arcs between Y and X).

(c) (5 points) Show that, if neither of the two above reduction rules is applicable and there are more than $k(k+2)$ vertices left, then the instance is a no-instance. Hence, the problem ARC FLIPPING IN TOURNAMENTS has a polynomial kernel.

> **Solution:** Suppose we have a yes-instance. Let A' be the set of arcs that is reversed in specific solution. Each arc $a \in A'$ is in at most k triangles because otherwise the first reduction rule would apply. Each of these triangles contains one vertex in addition to the two endpoints of a. Hence, there are at most $k+2$ vertices in a triangle containing a. Hence, in total, there are at most $k(k+2)$ vertices in a triangle containing an arc of A'. There can't be any other vertices because each vertex is incident to an arc of a triangle (otherwise the second reduction rule would apply) and each triangle contains an arc of A' (otherwise A' would not be a valid solution).

End of test