Exam WI4410 Advanced Discrete Optimization

August 15th, 2018, 13:30–16:30

The exam consists of 7 questions. In total you can obtain 60 points. Your grade is calculated by dividing the number of points you obtained by 6. You may use a non-graphical calculator during the exam. Using a graphical calculator, notes, phone, smart-watch, etc. is **not** permitted. The total number of pages of this exam is 7. Good luck!

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1. (a) (3 points) Consider the following set $X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 0 \leq x_i \leq 1, i = 1, \ldots, 2\}$. Give a valid inequality that defines:
   (i) an inproper face,
   (ii) a zero-dimensional face,
   (iii) a facet.

   Solution: Here is an example of a possible answer:

   (i) $x_1 \leq 2$,

   (ii) $x_1 + x_2 \leq 2$,

   (iii) $x_2 \leq 1$.

   (b) (6 points) Given a set $\{\boldsymbol{x} \in \mathbb{Z}^n \mid \sum_{j=1}^n a_j x_j = a_0\}$ with $a_0 \notin \mathbb{Z}$, the Gomory fractional cut based on this set is:

$$\sum_{j=1}^{n} f_j x_j \geq f_0,$$

   where $f_j := a_j - \lfloor a_j \rfloor$ and $f_0 := a_0 - \lfloor a_0 \rfloor$.
   Similarly, the Gomory Mixed-Integer Cut based on the set $\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z}^n \times \mathbb{R}^P \mid \sum_{j=1}^n a_j x_j + \sum_{j=1}^p g_j y_j = a_0\}$ with $a_0 \notin \mathbb{Z}$, is

$$\sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} \frac{f_0(1 - f_j)}{1 - f_0} x_j + \sum_{g_j > 0} g_j y_j - \sum_{g_j < 0} \frac{f_0}{1 - f_0} g_j y_j \geq f_0$$

   Consider the following integer optimization problem:

$$\begin{aligned} \max\ & 4x_1 + 3x_2 \\ \text{subject to } & 2x_1 + x_2 \leq 11 \\ & -x_1 + 2x_2 \leq 6 \\ & x_1,\ x_2 \geq 0, \quad \text{integer} \end{aligned}$$

Let s_1 and s_2 be integer slack variables in the above constraints. After solving the LP-relaxation of the problem we obtain:

$$\begin{array}{ccccccc} z & & & +\frac{11}{5}s_1 & +\frac{2}{5}s_2 & = & \frac{133}{5} \\ & & x_2 & +\frac{1}{5}s_1 & +\frac{2}{5}s_2 & = & \frac{23}{5} \\ & x_1 & & +\frac{2}{5}s_1 & -\frac{1}{5}s_2 & = & \frac{16}{5} \end{array}$$

(i) (4 pts) Generate a Gomory fractional cut *and* a Gomory Mixed-Integer Cut from the last row of the system of equations above.

(ii) (2 pts) Which of the two inequalities is stronger? A brief motivation suffices.

Solution: (i):

$$f_0 = \frac{1}{5},\ f_{s_1} = \frac{2}{5},\ f_{s_2} = \frac{4}{5}.$$

Gomory fractional cut:

$$\frac{2}{5}s_1 + \frac{4}{5}s_2 \geq \frac{1}{5}, \quad \text{or } 2s_1 + 4s_2 \geq 1.$$

Gomory mixed-integer cut:

$$\frac{\frac{1}{5}\cdot\frac{3}{5}}{\frac{4}{5}}s_1 + \frac{\frac{1}{5}\cdot\frac{1}{5}}{\frac{4}{5}}s_2 \geq \frac{1}{5}$$

or

$$\frac{3}{4}s_1 + \frac{1}{4}s_2 \geq 1.$$

(ii): The two inequalities have the same right-hand side, and each of the coefficients in the left-hand side of the GMIC is smaller than or equal to the corresponding coefficient in the Gomory fractional cut. Therefore the GMIC is stronger. This also holds in general!

2. Consider the single-node flow problem:

$$S_{SNF} := \{(\boldsymbol{x},\ \boldsymbol{y}) \in \mathbb{R}^n_+ \times \{0,1\}^n \mid \sum_{j\in N} x_j = b,\ x_j \leq u_j y_j\}.$$

A set $C \subseteq N$ is called a flow cover if $\sum_{j\in C} u_j > b$. Let $\lambda := \sum_{j\in C} u_j - b$ and $(u_j - \lambda)^+ := \max(u_j - \lambda,\ 0)$.

(a) (5 points) Prove that the flow cover inequality

$$\sum_{j\in C} x_j + \sum_{j\in C}(u_j - \lambda)^+(1 - y_j) \leq b$$

is valid for S_{SNF}. (Hint: Consider the case where $y_j = 1$ for all $j \in C$ and then the case where an arbitrary arc $k \in C$ is closed.)

Solution: See Lecture 3, slide 33.

(b) (2 points) Derive a valid flow cover inequality for the following example:

$$S_{SNF} := \{(\boldsymbol{x},\ \boldsymbol{y}) \in \mathbb{R}^4_+ \times \{0,1\}^4 \mid \sum_{j=1}^{4} x_j = 66,$$

$$0 \le x_1 \le 30y_1,\ 0 \le x_2 \le 50y_2,\ 0 \le x_3 \le 20y_3,\ 0 \le x_4 \le 45y_4\,.$$

> **Solution:** Take for instance $C = \{2,3\}$. That gives $\lambda = 50 + 20 - 66 = 70 - 66 = 4$ and the flow cover inequality
>
> $$x_2 + x_3 + 46(1 - y_2) + 16(1 - y_3) \le 66.$$

3. (a) (3 points) Let $N := \{1, \dots, n\}$. Consider the following knapsack set:

$$S_K := \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{j \in N} a_j x_j \le b\}\,,$$

Let $C \subseteq N$ be such that $\sum_{j \in C} a_j > b$. The family of knapsack cover inequalities, $\sum_{j \in C} x_j \le |C| - 1$ is valid for S_K. Now, consider the the specific instance

$$S_K = \{x \in \{0,1\}^4 \mid 25x_1 + 20x_2 + 15x_3 + 10x_4 \le 44\}\,.$$

The knapsack cover inequality $x_2 + x_3 \le 1$ is valid for the set

$$\text{conv}(S_K \cap \{x \in \mathbb{R}^4 \mid x_4 = 1\})\,.$$

Apply maximal lifting to the variable x_4 and give the resulting valid inequality.

> **Solution:** Introduce x_4 in the inequality. Don't forget that x_4 is currently set equal to 1:
>
> $$x_2 + x_3 + \beta x_4 \le 1 + \beta\,.$$
>
> Now set x_4 equal to 0 and apply maximal lifting
>
> $$\beta = \max\{x_2 + x_3 \mid 25x_1 + 20x_2 + 15x_3 \le 44\} - 1\,,$$
>
> which yields $\beta = 1$ and the resulting inequality $x_2 + x_3 + x_4 \le 2$.

(b) (1 point) Illustrate by a 2-dimensional example why branch-and-bound is not a polynomial-time algorithms in fixed dimension.

> **Solution:** See slide 31 of Lecture 4.

4. Consider the quadratic assignment problem $QAP(A,B,C)$:

$$z^* = \min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)} + \sum_{i=1}^n c_{i\varphi(i)},$$

where $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ are real $n \times n$ matrices, and $\mathcal{S}_n$ denotes the set of all permutations of $\{1, \ldots, n\}$. If C is the zero matrix, we write $QAP(A,B)$ instead of $QAP(A,B,C)$.

(a) (3 points) Given a graph $G = (V,E)$ with $|V|$ even, the maximum bisection problem is to partition V into two equal sets, say $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and $|V_1| = |V_2|$, so that the number of edges connecting a vertex in V_1 with a vertex in V_2 is a maximum. Explain how that problem may be rewritten as a problem of the form $QAP(A,B)$, i.e. formulate suitable matrices A and B in terms of G. Also explain how the partition (V_1, V_2) is obtained from the solution of the quadratic assignment problem you formulated.

Solution:
A is the adjacency matrix of G and $B = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J$, where J is the all-ones matrix of order $\frac{1}{2}|V|$. The vertices in V_1 are given by the labels $\varphi(i)$ $i \in \{1, \ldots, \frac{1}{2}|V|\}$ if φ is the solution of $QAP(A,B)$.

(b) (3 points) Prove that one may assume without loss of generality that the matrices A and B have zero diagonals. In other words, given matrices A, B, C, construct new matrices, say $\hat{A}$, $\hat{B}$, and $\hat{C}$ so that the diagonal elements of $\hat{A}$, $\hat{B}$ are zero, and $QAP(\hat{A}, \hat{B}, \hat{C})$ has the same objective function as $QAP(A,B,C)$.

Solution: Replace given A and B by the matrices $\hat{A}$ and $\hat{B}$ obtained by setting their respective diagonals to zero, and replace the given C by $\hat{C} = (c_{ij} + a_{ii} b_{jj})$.
Then $QAP(A,B,C)$ and $QAP(\hat{A}, \hat{B}, \hat{C})$ have the same objective function, since:

$$\begin{aligned} z(\varphi) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)} + \sum_{i=1}^n c_{i\varphi(i)} \\ &= \sum_{i \neq j} a_{ij} b_{\varphi(i)\varphi(j)} + \sum_{i=1}^n \left(c_{i\varphi(i)} + a_{ii} b_{\varphi(i)\varphi(i)} \right), \end{aligned}$$

and the last expression is exactly the objective function of $QAP(\hat{A}, \hat{B}, \hat{C})$.

(c) (4 points) Consider the objective function of $QAP(A,B,C)$, namely

$$z(\varphi) := \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\varphi(i)\varphi(j)} + \sum_{i=1}^n c_{i\varphi(i)}.$$

Prove that the average objective function value (taken over all $\varphi \in \mathcal{S}_n$) equals

$$\mu(A,B,C) = \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \sum_{\substack{k,l=1 \\ k \neq l}}^n a_{ij} b_{kl} + \frac{1}{n} \sum_{i,j=1}^n \left(c_{ij} + a_{ii} b_{jj} \right).$$

Hint: use the result of the previous exercise.

Solution: Applying Proposition 7.7 in the book to $QAP(\hat{A}, \hat{B}, \hat{C})$ (from the previous exercise) yields the required result.

5. We again consider $QAP(A, B, C)$ with objective function again denoted by

$$z(\varphi) := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\varphi(i)\varphi(j)} + \sum_{i=1}^{n} c_{i\varphi(i)}.$$

(a) (4 points) Assume that at some node in a polytomic branch-and-bound tree, the values $\varphi(i)$ $(i \in S)$ have been fixed for some $S \subset \{1, \dots, n\}$. Give the resulting QAP problem $QAP(A', B', C')$ at this node, i.e. give the matrices A', B' and C' in terms of A, B, C and S.

Solution: $QAP(A, B, C)$:

$$min_{\varphi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i=1}^{n} c_{i\varphi(i)}$$

Split the sums in the objective over S and its complement, say $\bar{S}$:

$$\begin{aligned} & \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i=1}^{n} c_{i\varphi(i)} \\ = \; & \sum_{i \in \bar{S}} \sum_{k \in \bar{S}} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i \in \bar{S}}^{n} c_{i\varphi(i)} \\ & + \sum_{i \in S} \sum_{k \in S} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i \in S}^{n} c_{i\varphi(i)} \\ & + \sum_{i \in S} \sum_{k \in \bar{S}} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i \in \bar{S}} \sum_{k \in S} a_{ik} b_{\varphi(i)\varphi(k)} \end{aligned}$$

Thus we get a new QAP, say $QAP(A', B', C')$ with $A' = (a'_{ij})$, $B' = (b'_{ij})$, $C' = (c'_{ij})$, and

$$c'_{ik} = \sum_{j \in S} (a_{ij} b_{kj} + a_{ji} b_{jk}) + c_{ik} \quad i, k \in \bar{S},$$

and

$$a'_{ij} = a_{ij}, \; b'_{ij} = b_{ij} \quad i, j \in \bar{S},$$

and a constant part

$$\text{const} = \sum_{i \in S} \sum_{k \in S} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i \in S}^{n} c_{i\varphi(i)}.$$

Thus we get:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{\varphi(i)\varphi(k)} + \sum_{i=1}^{n} c_{i\varphi(i)} \\ = \;& \text{const} + \sum_{i,k\in\bar{S}} a'_{ik} b'_{\varphi(i)\varphi(k)} + \sum_{i\in\bar{S}} c'_{i\varphi(i)}.\end{aligned}$$

(b) (4 points) Assume $A = uu^T$ and $B = vv^T$ for nonnegative vectors $u, v \in \mathbb{R}^n_+$. Prove that $QAP(A, B)$ may be solved in polynomial time in this case.

Solution: Proposition 8.9 in the book.

(c) (2 points) Use any method of your choice to solve the following instance of $QAP(A, B)$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix},\ B = \begin{pmatrix} 8 & 12 & 16 \\ 12 & 18 & 24 \\ 16 & 24 & 32 \end{pmatrix}.$$

Give the optimal value as well as the optimal permutation.

Solution: $A = uu^T$ with $u = [1\ 2\ 3]^T$, $B = vv^T$ with $v = \sqrt{2}[2\ 3\ 4]'$. Therefore, by Proposition 8.9, $z^* = (< u, v >^-)^2 = 2(1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2)^2 = 2 \cdot 16^2 = 512$, and $\varphi^* = (3, 2, 1)$.

6. (5 points) Consider the following parameterized problem.

RECOLOURING
Instance: graph $G = (V, E)$, a colouring $f : V \to \{1, 2, 3\}$ of its vertices with three colours and an integer $k \in \mathbb{Z}$.
Parameter: k.
Question: is it possible to obtain a proper 3-colouring (i.e. that $f(u) \neq f(v)$ for all $\{u, v\} \in E$) by changing the colour of at most k vertices?

Prove that the RECOLOURING problem is FPT. Also analyze the running time of your algorithm.

Solution: If there is no edge $\{u, v\} \in E$ with $f(u) = f(v)$ then we are done. Otherwise, choose such an edge and branch into four subproblems, in each subproblem, the colour of one of u, v is changed to one of the two remaining colours and the parameter is reduced by one. Since the search tree has depth at most k, the running time is $O(4^k|E|)$.

7. Consider the following parameterized problem.

EDGE CLIQUE COVER
Instance: graph $G = (V, E)$ and an integer $k \in \mathbb{Z}$.

Parameter: k.
Question: is it possible to cover all edges of G with at most k cliques?

Here, a *clique* is a subset $U \subseteq V$ such that $\{u, v\} \in E$ for all $u, v \in U$. An edge $\{u, v\} \in E$ is *covered* by a clique U if $u, v \in U$.

(a) (5 points) Consider the following reduction rule. If there is an edge $\{u, v\} \in E$ such that neither u nor v has any other neighbours, then delete u and v and reduce k by one. Show that this reduction rule is safe, i.e. that the original instance is a yes-instance if and only if the reduced instance is a yes-instance.

> **Solution:** Given at most $k - 1$ cliques that cover all edges of the reduced instance, we can add a clique $\{u, v\}$ and obtain at most k cliques that cover all edges of the original instance. Conversely, if there exist at most k cliques that cover all edges of the original instance, then $\{u, v\}$ must be one of these cliques because this is the only possible clique that covers the edge $\{u, v\}$. Removing $\{u, v\}$ from the set of cliques then gives at most $k - 1$ cliques that cover all edges of the reduced instance.

(b) (5 points) Consider the following reduction rule. If there is an edge $\{u, v\} \in E$ such that u and v have exactly the same set of neighbours (but the first reduction rule does not apply), then delete exactly one of u and v, without changing k. Show that this reduction rule is safe.

> **Solution:** It is clear that if the original instance is a yes-instance then the reduced instance is a yes-instance. To show the converse, suppose there exist at most k cliques that cover all edges of the reduced instance and suppose that vertex u was deleted by the reduction rule. Then we can add u to each clique containing v (there must be at least one because otherwise the first reduction rule would have been applicable) and then the cliques cover all edges of the original instance.

(c) (5 points) A third reduction rule is to delete any isolated vertices. Show that if none of these three reduction rules is applicable and there are more than 2^k vertices left then the instance is a no-instance. Hence, EDGE CLIQUE COVER has a kernel with at most 2^k vertices.

> **Solution:** Suppose the instance is a yes-instance. Then there exist at most k cliques that cover all vertices. Since there are more than 2^k vertices, there exist two vertices u, v that are in exactly the same subset of the cliques (by the pigeonhole principle since there are at most 2^k subsets of the set of cliques). This implies that u and v have exactly the same set of neighbours, contradicting our assumption that none of the reduction rules is applicable.

End of test