Solutions Exam Complex Function Theory AM2040

Friday July 24, 2020, 13:30–16:30

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1. Write the following sentence on your exam:

   > I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud.

2. Let $z^{\frac{1}{2}}$ be the square root function with branch cut $\{z \in \mathbb{C} \mid \text{Im}(z) = 0,\ \text{Re}(z) \geq 0\}$. Define

   $$F(z) = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}.$$

   Find the largest region on which F is analytic.

   F is the composition of $\frac{1+z}{1-z}$ and $z^{\frac{1}{2}}$, so F is analytic for all z where $\frac{1+z}{1-z}$ is analytic and where $\frac{1+z}{1-z}$ is in the region where $z^{\frac{1}{2}}$ is analytic.

   $\frac{1+z}{1-z}$ is not analytic at $z = 1$, so F also has a singularity at $z = 1$.

   Note that $z^{\frac{1}{2}}$ is analytic on $\mathbb{C} \setminus [0, \infty)$. We determine for which $z \in \mathbb{C}$ the number $\frac{1+z}{1-z}$ is on the branch cut $[0, \infty)$, so we solve the equation

   $$\frac{1+z}{1-z} = a, \qquad a \in [0, \infty).$$

   We have for $z \neq 1$

   $$1 + z = (1-z)a \quad \Leftrightarrow \quad (1+a)z = a - 1 \quad \Leftrightarrow \quad z = \frac{a-1}{a+1}.$$

   Since $\frac{a-1}{a+1} \in [-1, 1)$ for $a \in [0, \infty)$, we see that F is analytic on $\mathbb{C} \setminus [-1, 1]$.

3. Prove or disprove: There exists an analytic function $f : \mathbb{C} \to \mathbb{C}$ with

   $$\text{Re}(f(x+iy)) = x^3 y.$$

If such a function exists, then x^3y is a harmonic function, i.e.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)x^3y=0,$$

for all $(x,y)\in\mathbb{R}^2$. But we have

$$\frac{\partial^2}{\partial x^2}x^3y=6xy, \qquad \frac{\partial^2}{\partial y^2}x^3y=0,$$

so the function is not harmonic. We conclude that x^3y is not the real part of an analytic function.

Alternative: Suppose that such a function f exists, then f satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

where $u(x,y)=\text{Re}(f(x+iy))$ and $v(x,y)=\text{Im}(f(x+iy))$. So $u(x,y)=x^3y$, and

$$\frac{\partial v}{\partial y}=3x^2y,$$

so that $v(x,y)=\frac{3}{2}x^2y^2+C_1(x)$. Also

$$\frac{\partial v}{\partial x}=-x^3,$$

so that $v(x,y)=-\frac{1}{4}x^4+C_2(y)$. Then

$$\frac{3}{2}x^2y^2+C_1(x)=-\frac{1}{4}x^4+C_2(y),$$

for all $(x,y)\in\mathbb{R}^2$, or

$$C_2(y)=\frac{3}{2}x^2y^2+\frac{1}{4}x^4+C_1(x),$$

so that C_2 is not independent of x. We conclude that there is no analytic function f with $\text{Re}(f(x,y))=x^3y$.

4. Let g be an entire function satisfying

$$|g(z)|\leq\ln(|z|+1), \qquad z\in\mathbb{C}.$$

Show that g is constant.

Let $z_0 \in \mathbb{C}$. Since g is analytic on $B_R(z_0)$ for arbitrary $R > 0$, we can use the Cauchy estimate for f' on $B_R(z_0)$. Let $z \in B_R(z_0)$, then $|z| \leq |z_0| + R$. We have

$$|g(z)| \leq \ln(|z| + 1) \leq \ln(|z_0| + R + 1),$$

so

$$|g'(z_0)| \leq \frac{\ln(|z_0| + R + 1)}{R}.$$

Since

$$\lim_{R\to\infty} \frac{\ln(|z_0| + R + 1)}{R} = 0,$$

it follows that $g'(z_0) = 0$ for any $z_0 \in \mathbb{C}$. This implies that g is a constant function.

Alternative: Let $z \in \mathbb{C}$. Since g is analytic on $\mathbb{C}$ the generalized Cauchy integral formula says that

$$g'(z) = \frac{1}{2\pi i} \int_{C_R(0)} \frac{g(w)}{(w - z)^2}\, dw, \qquad \text{for } R > |z|$$

Using the ML-inequality we obtain

$$|g'(z)| \leq \frac{1}{2\pi} \left| \int_{C_R(0)} \frac{g(w)}{(w - z)^2}\, dw \right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \max_{|w|=R} \left| \frac{g(w)}{(w - z)^2} \right| \leq R \cdot \frac{\ln(R + 1)}{(R - |z|)^2}.$$

This holds for all $R > |z|$, so letting $R \to \infty$ shows that that $|g'(z)| = 0$. Then $g'(z) = 0$ for all $z \in \mathbb{C}$, hence g is constant.

5. Determine and classify the isolated singularities in $\mathbb{C}$ (not ∞) of

$$h(z) = \frac{\cosh(\frac{i\pi}{z}) + 1}{z^2 + 1}.$$

First determine the zeros of the denominator:

$$z^2 + 1 = 0 \quad \Rightarrow \quad z = i,\ z = -i.$$

Note that $\cos(\frac{i\pi}{\pm i}) = \cos(\pm\pi) = -1$. Using l'Hôpital's rule we find

$$\lim_{z\to\pm i} \frac{\cos(\frac{i\pi}{z}) + 1}{z^2 + 1} = \lim_{z\to\pm i} \frac{-\sin(\frac{i\pi}{z}) \cdot \frac{-i\pi}{z^2}}{2z} = \frac{i\pi \sin(\pm\pi i)}{2(\pm i)^3} = 0,$$

so $z = i$ and $z = -i$ are removable singularities. Furthermore, there is an essential singularity at 0: Letting $z \to 0$ over the real axis gives

$$\lim_{x\to 0} \cos(\frac{i\pi}{x}) = \lim_{x\to 0} \frac{1}{2}(e^{\frac{\pi}{x}} + e^{-\frac{\pi}{x}}) = \infty,$$

so

$$\lim_{x\to 0}\frac{\cos(\frac{i\pi}{x})+1}{x^2+1}=\infty$$

showing that $z=0$ is a nonremovable singularity. For $z=iy$ on the imaginary axis and $y\in(-\frac{1}{2},\frac{1}{2})$ we have

$$|h(iy)|=\left|\frac{\cos(\frac{\pi}{y})+1}{1-y^2}\right|\leq\frac{1+1}{1-\frac{1}{4}},$$

showing that $z=0$ is not a pole (if it was a pole, then $\lim_{z\to 0}|h(z)|=\infty$).

Alternatives:

$$\lim_{z\to\pm i}(z\mp i)h(z)=\lim_{z\to\pm i}\frac{\cos(\frac{i\pi}{z})+1}{z\pm i}=\frac{\cos(\pm\pi)+1}{\pm 2i}=0,$$

so $z=\pm i$ is a removable singularity.

The Laurent series at 0 for $\cos(\frac{i\pi}{z})+1$ is

$$2+\sum_{n=1}^{\infty}\frac{\pi^{2n}}{(2n)!}\frac{1}{z^{2n}},$$

and for $\frac{1}{z^2+1}$

$$\sum_{n=0}^{\infty}(-1)^n z^{2n}.$$

Multiplying these two series shows that the Laurent series for h at 0 contains infinitely many nonzero coefficients for negative powers of z, so $z=0$ is an essential singularity.

6. Determine the Laurent series expansion of the following functions on the region $\{z\in\mathbb{C}\,:\,|z+i|>4\}$.

a) $\dfrac{1}{iz+3}$

b) $\dfrac{1}{(iz+3)^2}$

a) Use the geometric series:

$$\frac{1}{iz+3}=\frac{1}{i(z+i)+4}=\frac{1}{i(z+i)}\frac{1}{1-\frac{4i}{z+i}}=\frac{1}{i(z+i)}\sum_{n=0}^{\infty}\frac{(4i)^n}{(z+i)^n}=-i\sum_{n=1}^{\infty}\frac{(4i)^{n-1}}{(z+i)^n},$$

which converges absolutely for $|\frac{4i}{z+i}|<1$, i.e. $|z+i|>4$.

b) Use

$$\frac{d}{dz}\frac{1}{iz+3} = \frac{-i}{(iz+3)^2},$$

then using part a we obtain

$$\frac{1}{(iz+3)^2} = i\frac{d}{dz}\frac{1}{iz+3} = \sum_{n=1}^{\infty}(4i)^{n-1}\frac{d}{dz}(z+i)^{-n} = \sum_{n=1}^{\infty}\frac{(4i)^{n-1}n}{(z+i)^{n+1}}.$$

We may interchange the order of differentiation and summation because the series converges uniformly on any closed set within the region of convergence. The region of convergence does not change, so this is still valid for $|z+i| > 4$.

7. Evaluate the improper integral

$$\int_{-\infty}^{\infty}\frac{\sin(x)}{x^2+2x+2}\,dx.$$

We integrate the function

$$f(z) = \frac{e^{iz}}{z^2+2z+2}$$

over the closed path C consisting of $I_R = [-R, R]$ and the arc $C_R = \{Re^{i\theta} \mid 0 \le \theta \le \pi\}$, oriented in the counterclockwise direction. The function f is analytic on $\mathbb{C}$ except at the zeros of the denominator:

$$z^2+2z+2 = 0 \quad\Leftrightarrow\quad (z+1)^2+1 = 0 \quad\Leftrightarrow\quad (z+1)^2 = -1 \quad\Leftrightarrow\quad z = -1 \pm i.$$

Only $-1+i$ is inside C. Now we calculate the integral over C using the residue theorem,

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\,\mathrm{Res}(f, -1+i)\\ &= 2\pi i \lim_{z\to -1+i}(z+1-i)\frac{e^{iz}}{(z+1-i)(z+1+i)}\\ &= 2\pi i\frac{e^{-i-1}}{2i} = \pi e^{-i-1}.\end{aligned}$$

Next we show that the integral over the arc C_R vanishes as $R \to \infty$. Note that for $z \in C_R$

$$|e^{iz}| = |e^{iR(\cos\theta + i\sin\theta)}| = e^{-R\sin\theta} \le 1.$$

Using the ML-inequality we obtain

$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R \cdot \frac{1}{R^2-2R-2},$$

so that

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

We conclude that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+2x+2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2+2x+2}\,dx$$
$$= \lim_{R\to\infty} \int_{I_R} f(z)\,dz$$
$$= \lim_{R\to\infty} \int_C - \int_{C_R} f(z)\,dz = \pi e^{-i-1}.$$

Taking the imaginary part gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+2x+2}\,dx = -\pi e^{-1}\sin(1).$$

It remains to show that the principal value integral is equal to the improper integral. We have for large $|x|$

$$\left|\frac{\sin(x)}{x^2+2x+2}\right| \le \frac{2}{x^2},$$

and the integrals $\int_1^{\infty} \frac{2}{x^2}\,dx$ and $\int_{-\infty}^{-1} \frac{2}{x^2}\,dx$ converge. So the improper integral converges, and then

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+2x+2}\,dx = \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+2x+2}\,dx = -\pi e^{-1}\sin(1).$$

8. a) Show that $z^{2020}+z+1$ has all its zeros in $\{z \in \mathbb{C} : |z| < 2\}$.

   b) Evaluate

$$\int_{C_2(0)} \frac{z^{2019}}{z^{2020}+z+1}\,dz,$$

   where $C_2(0)$ has positive orientation.
   Hint: consider the integral over $C_R(0)$ with $R > 2$ and use part a.

   a) Let $f(z) = z^{2020}$ and $g(z) = z+1$. Then for $|z| = 2$, we have

$$|g(z)| \le |z| + 1 = 3 \le 2^{2020} = |f(z)|.$$

   Both f and g are analytic on and insdie $C_2(0)$, so the number of zeros of $f+g$ on $B_2(0)$ is equal to the number of zeros of f on $B_2(0)$ by Rouché's theorem. Clearly the only zero of f is 0 with multiplicity 2020, so $f+g$ also has 2020 zeros (counted with multiplicity) in $B_2(0)$. Since $f+g$ is a polynomial of degree 2020 there are exactly 2020 zeros by the fundamental theorem of algebra, so *all* zeros are in $B_2(0)$.

b) By the counting theorem and part a we have

$$\int_{C_2(0)} \frac{2020z^{2019}+1}{z^{20202}+z+1}dz = 2\pi i \cdot 2020,$$

so that

$$\int_{C_2(0)} \frac{2020z^{2019}}{z^{20202}+z+1}dz = 2\pi i \cdot 2020 - \int_{C_2(0)} \frac{1}{z^{20202}+z+1}dz.$$

Since $\frac{1}{z^{2020}+z+1}$ has no singularities on the outside of $C_2(0)$, we have

$$\int_{C_2(0)} \frac{1}{z^{20202}+z+1}dz = \int_{C_R(0)} \frac{1}{z^{20202}+z+1}dz, \qquad R > 2,$$

by Cauchy's theorem for multiply connected regions. Using the ML-inequality we find

$$\left|\int_{C_R(0)} \frac{1}{z^{20202}+z+1}dz\right| \le 2\pi R \cdot \frac{1}{R^{2020}-R-1},$$

so that this integral vanishes as $R \to \infty$. We conclude that

$$\int_{C_2(0)} \frac{z^{2019}}{z^{20202}+z+1}dz = 2\pi i \frac{2020}{2020} = 2\pi i.$$