Solutions Exam Complex Function Theory AM2040

Monday June 29, 2020, 13:30–16:30

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(0) 1. Write the following sentence on your exam:

> I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud.

(10) 2. Let the function L be a branch of the logarithm with branch cut

$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) = 0,\ \mathrm{Im}(z) \geq 0\}.$$

Evaluate

$$\int_{[iR,-1+iR,-i,1+iR,iR]} L(z)\, dz, \qquad R > 0.$$

We consider

$$\int_{[iR-\varepsilon,-1+iR,-i,1+iR,iR+\varepsilon]} L(z)\, dz$$

where we let $\varepsilon \downarrow 0$. For L we can choose for example $\log_{-\frac{3}{2}\pi}$. Then L is analytic on $\mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 0,\ \mathrm{Im}(z) \geq 0\}$, so the integral only depends on the end points and not on the path itself. We integrate over the arc $C = \{Re^{i\theta} \mid -\frac{3}{2}\pi \leq \theta \leq \frac{1}{2}\pi\}$ in positive direction. On this arc we have

$$L(z) = \log_{-\frac{3}{2}\pi}(z) = \ln|z| + i\arg_{-\frac{3}{2}\pi}(z) = \ln R + i\theta.$$

Then

$$\begin{aligned}
\int_C L(z)\, dz &= \int_{-\frac{3}{2}\pi}^{\frac{1}{2}\pi} (\ln R + i\theta) \cdot iRe^{i\theta}\, d\theta \\
&= 0 - R\int_{-\frac{3}{2}\pi}^{\frac{1}{2}\pi} \theta e^{i\theta}\, d\theta \\
&= -R\Big[-i\theta e^{i\theta}\Big]_{-\frac{3}{2}\pi}^{\frac{1}{2}\pi} - iR\int_{-\frac{3}{2}\pi}^{\frac{1}{2}\pi} e^{i\theta}\, d\theta \\
&= -R\left(\frac{1}{2}\pi + \frac{3}{2}\pi\right) = -2\pi R.
\end{aligned}$$

Alternative: use a primitive function of L and evaluate in the end points.

(10) 3. Determine and classify the isolated singularities of $f(z) = \dfrac{z+i\pi}{z\sinh^2(z)}$.

The zeros of the denominator are at $k\pi i$, $k \in \mathbb{Z}$.

For $k \neq 0, -1$:

$$\lim_{z\to k\pi i}(z-k\pi i)^2 f(z) = \frac{k\pi i + \pi i}{k\pi i}\lim_{z\to k\pi i}\left(\frac{z-k\pi i}{\sinh(z)}\right)^2 = \frac{k+1}{k} \neq 0,$$

so $k\pi i$ is a pole of order 2.

For $k = -1$:

$$\lim_{z\to -i\pi}(z+\pi i)f(z) = \frac{1}{-i\pi}\lim_{z\to -\pi i}\left(\frac{z+\pi i}{\sinh(z)}\right)^2 = \frac{i}{\pi} \neq 0,$$

so $-\pi i$ is a pole of order 1.

For $k = 0$:

$$\lim_{z\to 0} z^3 f(z) = i\pi \lim_{z\to 0}\left(\frac{z}{\sinh(z)}\right)^2 = i\pi \neq 0,$$

so 0 is a pole of order 3.

(10) 4. Show that $g(z) = \displaystyle\sum_{n=1}^{\infty}\frac{z^n}{1+z^n}$ is analytic on $\{z \in \mathbb{C} \mid |z| < 1\}$.

Let $E \subseteq B_1(0)$ be a closed set. There exists an $r \in (0,1)$ such that $E \subseteq \overline{B_r(0)}$. Then for $z \in E$

$$\left|\frac{z^n}{1+z^n}\right| \leq \frac{r^n}{1-r^n} \leq \frac{r^n}{1-r} =: M_n$$

and

$$\sum_{n=1}^{\infty} M_n = \frac{1}{1-r}\sum_{n=1}^{\infty} r^n$$

converges. By the Weierstrass M-test $\sum_{n=1}^{\infty}\frac{z^n}{1+z^n}$ converges uniformly on E. $\frac{z^n}{1+z^n}$ is analytic on $B_1(0)$, since it has its singularities on $|z| = 1$. Now $g(z)$ is a series of analytic functions that is uniform convergent on every closed set in $B_1(0)$, so g is analytic on $B_1(0)$.

(12) 5. Determine the Laurent series expansion of

$$h(z) = \frac{i}{(z+i)(z-3i)}$$

on the region $\{z \in \mathbb{C} \mid 1 < |z-2i| < 3\}$.

First determine the partial fraction decomposition for h:

$$\frac{i}{(z+i)(z-3i)} = \frac{A}{z+i} + \frac{B}{z-3i},$$

then $A+B=0$ and $B-3A=1$, so that $A=-\frac{1}{4}$ and $B=\frac{1}{4}$.
Using the geometric series we obtain

$$\frac{1}{z+i} = \frac{1}{(z-2i)+3i} = \frac{1}{3i}\frac{1}{1-\frac{z-2i}{-3i}} = \frac{-i}{3}\sum_{n=0}^{\infty}\left(\frac{i}{3}\right)^n (z-2i)^n, \qquad |z-2i|<3,$$

and

$$\frac{1}{z-3i} = \frac{1}{(z-2i)-i} = \frac{1}{z-2i}\frac{1}{1-\frac{i}{z-2i}} = \sum_{n=1}^{\infty} i^{n-1}(z-2i)^{-n}, \qquad |z-2i|>1.$$

Then

$$h(z) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{i}{3}\right)^{n+1}(z-2i)^n + \frac{1}{4}\sum_{n=1}^{\infty} i^{n-1}(z-2i)^{-n}.$$

for $1 < |z-2i| < 3$.

6. Let f be an entire function.

(10) a) Prove that $f^*(z) := \overline{f(\overline{z})}$ is also entire.

(10) b) Suppose $f(z) \in \mathbb{R}$ for $z \in (-2020, 2020)$. Show that

$$f(\overline{z}) = \overline{f(z)}, \qquad z \in \mathbb{C}.$$

a) Use the Cauchy-Riemann equations. If $f(x+iy) = u(x,y)+iv(x,y)$ and $f^*(x+iy) = u^*(x,y)+iv^*(x,y)$, then

$$f^*(x+iy) = \overline{f(x-iy)} = u(x,-y) - iv(x,-y),$$

so that $u^*(x,y) = u(x,-y)$ and $v^*(x,y) = -v(x,-y)$. Since f is analytic u and v satisfy the CR equations. Then using the chain rule

$$u_x^*(x,y) = u_x(x,-y) = v_y(x,-y) = v_y^*(x,y)$$

and

$$u_y^*(x,y) = -u_y(x,-y) = v_x(x,-y) = -v_x^*(x,-y).$$

So u^* and v^* also satisfy the CR-equations. Since u and v are (real) differentiable, so are u^* and v^*. Conclusion: f^* is also entire.

Alternative: Since f is entire it has a power series expansion at 0 with radius of convergence ∞,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then f^* has the power series expansion

$$f^*(z) = \overline{\sum_{n=0}^{\infty} a_n \overline{z}^n} = \sum_{n=0}^{\infty} \overline{a_n} z^n,$$

again with radius of convergence ∞, hence f^* is analytic on $\mathbb{C}$.

Yet another alternative: Let us denote complex conjugation by $c(z) := \overline{z}$. Note that c is continuous on $\mathbb{C}$. Let $z_0 \in \mathbb{C}$. If $\lim_{z\to z_0} h(z) = L$, then

$$\lim_{z\to z_0} \overline{h(z)} = \lim_{z\to z_0} c \circ h(z) = c(L) = \overline{L}.$$

Furthermore, if $\lim_{z\to \overline{z_0}} h(z) = L'$,

$$\lim_{z\to z_0} h(\overline{z}) = \lim_{z\to z_0} h \circ c(z) = \lim_{u\to c(z_0)} h(u) = L'.$$

Combining these two properties we have

$$\lim_{z\to z_0} \overline{h(\overline{z})} = \overline{L'}.$$

Now let

$$h(z) = \frac{f(z) - f(\overline{z_0})}{z - \overline{z_0}}.$$

Since f is analytic in $\overline{z_0}$ we have $L' = \lim_{z\to \overline{z_0}} h(z) = f'(\overline{z_0})$. Then

$$\begin{aligned}\lim_{z\to z_0} \frac{f^*(z) - f^*(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} \\ &= \lim_{z\to z_0} \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}}\right)} \\ &= \lim_{z\to z_0} \overline{h(\overline{z})} = \overline{f'(\overline{z_0})},\end{aligned}$$

so f^* is differentiable in z_0. This holds for any $z_0 \in \mathbb{C}$, so f^* is analytic on $\mathbb{C}$.

b) Use the Identity Principle. For $z \in E = (-2020, 2020)$,

$$f^*(z) = \overline{f(\overline{z})} \overset{z\in\mathbb{R}}{=} \overline{f(z)} \overset{f(z)\in\mathbb{R}}{=} f(z).$$

E has an accumulation point in $\mathbb{C}$, f and f^* are both analytic on $\mathbb{C}$ by part a, so by the identity principle $f(z) = f^*(z)$ for any $z \in \mathbb{C}$. So $f(\overline{z}) = \overline{f(z)}$.

(12) 7. Let $a \in \mathbb{R}$ with $|a| > 1$. Evaluate

$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos(\theta)}\, d\theta.$$

Substitute $z = e^{i\theta}$, then

$$\begin{aligned} I &= \int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos(\theta)}\, d\theta \\ &= \int_{C_1(0)} \frac{1}{1 + a^2 - a(z + z^{-1})} \frac{dz}{iz} \\ &= \frac{i}{a} \int_{C_1(0)} \frac{1}{z^2 - (a + a^{-1})z + 1}\, dz \\ &= \frac{i}{a} \int_{C_1(0)} \frac{1}{(z-a)(z-a^{-1})}\, dz, \end{aligned}$$

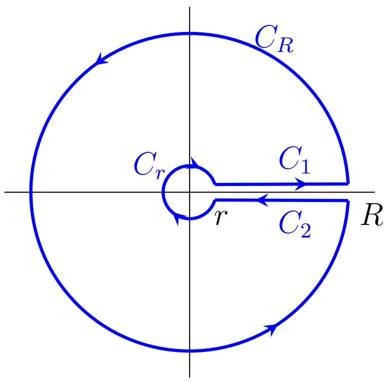
where $C_1(0)$ has positive orientation. The integrand $f(z) = \frac{1}{(z-a)(z-a^{-1})}$ is analytic in an on $C_1(0)$, except for a simple pole at a^{-1}. Then using the Residue Theorem

$$I = 2\pi i \frac{i}{a} \operatorname{Res}(f, a^{-1}) = -\frac{2\pi}{a} \lim_{z\to a^{-1}} (z - a^{-1}) f(z) = \frac{2\pi}{a^2 - 1}.$$

(16) 8. Evaluate

$$\int_0^{\infty} \frac{x^{\frac{1}{3}}}{x^2 + x + 1}\, dx$$

using the path as indicated in the picture.



We evaluate

$$I = \int_C \frac{z^{\frac{1}{3}}}{z^2 + z + 1}\, dz,$$

where C is the path in the picture, and we use the branch of $z^{1/3}$ with branch cut $[0,\infty)$, i.e. $z^{\frac{1}{3}} = e^{\frac{1}{3}\log_0(z)}$.

The integrand $f(z) = \frac{z^{\frac{1}{3}}}{z^2+z+1}$ has singularities at the zeros of the denominator, so at

$$z_1 = -\frac{1}{2} + i\frac{1}{2}\sqrt{3} = e^{\frac{2}{3}\pi i}, \qquad z_2 = -\frac{1}{2} - i\frac{1}{2}\sqrt{3} = e^{\frac{4}{3}\pi i}.$$

These are both poles of order 1, and they are both inside the path C (for small r and large R). The function f is analytic in and on C, except at its poles, so by the Residue Theorem

$$I = 2\pi i\left(\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2)\right).$$

We have

$$\mathrm{Res}(f, z_1) = \lim_{z\to z_1}(z - z_1)f(z) = \lim_{z\to e^{\frac{2}{3}\pi i}} \frac{z^{\frac{1}{3}}}{z - e^{\frac{4}{3}\pi}} = \frac{e^{\frac{2}{9}\pi i}}{e^{\frac{2}{3}\pi i} - e^{\frac{4}{3}\pi i}}$$

and

$$\mathrm{Res}(f, z_2) = \lim_{z\to z_2}(z - z_2)f(z) = \lim_{z\to e^{\frac{2}{3}\pi i}} \frac{z^{\frac{1}{3}}}{z - e^{\frac{4}{3}\pi}} = \frac{e^{\frac{4}{9}\pi i}}{e^{\frac{4}{3}\pi i} - e^{\frac{2}{3}\pi i}},$$

so that

$$I = 2\pi i\frac{e^{\frac{4}{9}\pi i} - e^{\frac{2}{9}\pi i}}{e^{\frac{4}{3}\pi i} - e^{\frac{2}{3}\pi i}} = -\frac{2\pi}{\sqrt{3}}(e^{\frac{4}{9}\pi i} - e^{\frac{2}{9}\pi i}).$$

For the integral over C_R we obtain from the ML-inequality:

$$\left|\int_{C_R} \frac{z^{\frac{1}{3}}}{z^2+z+1}\,dz\right| \leq 2\pi R\frac{R^{\frac{1}{3}}}{R^2 - R - 1},$$

so that

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{\frac{1}{3}}}{z^2+z+1}\,dz = 0.$$

Similarly for the integral over C_r:

$$\left|\int_{C_r} \frac{z^{\frac{1}{3}}}{z^2+z+1}\,dz\right| \leq 2\pi r\frac{r^{\frac{1}{3}}}{1 - r - r^2},$$

so that

$$\lim_{r\downarrow 0}\int_{C_r} \frac{z^{\frac{1}{3}}}{z^2+z+1}\,dz = 0.$$

The integral over C_2 and C_1 can be rewritten as

$$\int_{C_1} \frac{z^{\frac{1}{3}}}{z^2+z+1}\,dz = \int_r^R \frac{x^{\frac{1}{3}}}{x^2+x+1}\,dx$$

and

$$\int_{C_2} \frac{z^{\frac{1}{3}}}{z^2+z+1}\,dz = \int_R^r \frac{(e^{2\pi i}x)^{\frac{1}{3}}}{x^2+x+1}\,dx = -e^{\frac{2}{3}\pi i}\int_r^R \frac{x^{\frac{1}{3}}}{x^2+x+1}\,dx.$$

Combining all gives

$$(1-e^{\frac{2}{3}\pi i})\int_0^\infty \frac{x^{\frac{1}{3}}}{x^2+x+1}\,dx = I - \lim_{R\to\infty}\int_{C_R} f(z)dz - \lim_{r\downarrow 0}\int_{C_r} f(z)dz,$$

so that

$$\int_0^\infty \frac{x^{\frac{1}{3}}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}\frac{e^{\frac{2}{9}\pi i}-e^{\frac{4}{9}\pi i}}{1-e^{\frac{2}{3}\pi i}} = \frac{2\pi}{\sqrt{3}}\frac{e^{-\frac{1}{9}\pi i}-e^{\frac{1}{9}\pi i}}{e^{-\frac{1}{3}\pi i}-e^{\frac{1}{3}\pi i}} = \frac{4}{3}\pi\sin(\frac{1}{9}\pi).$$