Solutions Midterm Assignments Complex Function Theory, AM2040
May 2020

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1. Determine all solutions of the equation $e^{z^2} = 1$.

   Use $1 = e^{2k\pi i}$ with $k \in \mathbb{Z}$, then we see that

$$e^{z^2} = 1 \qquad \Leftrightarrow \qquad z^2 = 2k\pi i, \quad k \in \mathbb{Z}.$$

   There are three cases to consider: $k > 0$, $k < 0$ and $k = 0$.

$k = 0$: $z^2 = 0$ has solution $z = 0$.

$k > 0$: $z^2 = 2k\pi i = 2k\pi e^{\frac{1}{2}\pi i + 2n\pi i}$ with $n \in \mathbb{Z}$. Let $z = re^{i\theta}$, then $z^2 = r^2 e^{2i\theta}$, so that

$$r = \sqrt{2k\pi}, \qquad \theta = \frac{1}{4}\pi, \frac{5}{4}\pi.$$

So we find the solutions

$$z = \sqrt{2k\pi}\left(\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}\right) = \sqrt{k\pi}(1+i),$$

and

$$z = \sqrt{2k\pi}\left(-\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\right) = -\sqrt{k\pi}(1+i).$$

$k < 0$: $z^2 = -2k\pi(-i) = -2k\pi e^{-\frac{1}{2}\pi i + 2n\pi i}$ with $n \in \mathbb{Z}$. Let $z = re^{i\theta}$, then $z^2 = r^2 e^{2i\theta}$, so that

$$r = \sqrt{-2k\pi}, \qquad \theta = -\frac{1}{4}\pi, \frac{3}{4}\pi.$$

So we find the solutions

$$z = \sqrt{-2k\pi}\left(\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\right) = \sqrt{-k\pi}(1-i),$$

and

$$z = \sqrt{-2k\pi}\left(-\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}\right) = -\sqrt{-k\pi}(1-i).$$

2. Compute the limit or show that the limit does not exist.

   a) $\lim\limits_{z\to\infty} e^{-(z+i)^2}$

   Let $z \to \infty$ along the line $x - i$ ($x \in \mathbb{R}$), then

$$\lim_{x\to\infty} e^{-((x-i)+i)^2} = \lim_{x\to\infty} e^{-x^2} = 0.$$

   Then let $z \to \infty$ along the line $i(y-1)$ (the imaginary axis), then

$$\lim_{y\to\infty} e^{-((iy-i)+i)^2} = e^{y^2} = \infty.$$

   So the limit does not exist.

b) $\lim_{z\to 0} z \operatorname{Log}(z-1)$.

Note that $\lim_{z\to 0} \operatorname{Log}(z-1)$ does not exist, so we cannot use the rule $\lim f(z)g(z) = \lim f(z) \cdot \lim g(z)$. We use the fact that $\operatorname{Log}(z-1)$ is bounded on a disc around 0: for $r > 0$ and $z \in B_r(0)$

$$|\operatorname{Log}(z-1)|^2 = (\ln|z-1|)^2 + (\operatorname{Arg}(z-1))^2 \leq (\ln|1+r|)^2 + \pi^2.$$

Then since $\lim_{z\to 0} z = 0$, we have

$$\lim_{z\to 0} z \operatorname{Log}(z-1) = 0.$$

3. Let $f(z) = z \operatorname{Re}(z)$. Determine all points $z_0 \in \mathbb{C}$ for which the complex derivative $f'(z_0)$ exists.

We use the definition of the derivative. Let $z_0 \in \mathbb{C}$. For $h \in \mathbb{C}$ let $DQ(h)$ be the difference quotient given by

$$DQ(h) = \frac{(z_0+h)\operatorname{Re}(z_0+h) - z_0 \operatorname{Re}(z_0)}{h} = \frac{z_0 \operatorname{Re}(h) + h \operatorname{Re}(z_0) + h \operatorname{Re}(h)}{h},$$

then f is differentiable at z_0 iff $\lim_{h\to 0} DQ(h)$ exists.

Let $h = t$ with $t \in \mathbb{R}$. Then

$$\lim_{t\to 0} DQ(t) = \lim_{t\to 0} \frac{z_0 t + t\operatorname{Re}(z_0) + t^2}{t} = z_0 + \operatorname{Re}(z_0).$$

Let $h = it$ with $t \in \mathbb{R}$. Then

$$\lim_{t\to 0} DQ(it) = \lim_{t\to 0} \frac{it \operatorname{Re}(z_0)}{it} = \operatorname{Re}(z_0).$$

These limits are only equal if $z_0 = 0$, so we may conclude that $z \operatorname{Re}(z)$ is not complex differentiable at any point $z_0 \neq 0$. At $z_0 = 0$ the complex derivative does exist:

$$\lim_{h\to 0} \frac{h \operatorname{Re}(h) - 0}{h} = \lim_{h\to 0} \operatorname{Re}(h) = 0.$$

4. Let f be an analytic function on a region Ω satisfying

$$f(z) = u(x) + iv(y), \qquad z = x + iy \in \Omega,$$

where u and v are real functions. Show that $f(z) = az + b$ for certain constants $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

From the Cauchy-Riemann equations we find $u'(x) = v'(y)$ for all $x, y \in \mathbb{R}$, which implies both are equal to a real constant a. Then $u(x) = ax + c$ and $v(y) = ay + d$, so $f(x+iy) = a(x+iy) + b$ with $b = c + id$.

5. Calculate the following integrals.

a) $\displaystyle\int_{[z_0,z_1,z_2,z_3]} (z-1)^{\frac{1}{2}}\,dz$, where $z_0 = 1+i$, $z_1 = 2+3i$, $z_2 = 4-i$, $z_3 = 1-i$ and we use the principal branch of the power function.

The function $f(z) = (z-1)^{\frac{1}{2}}$ is analytic on $\mathbb{C} \setminus (-\infty, 1]$, and the given path lies in this region. Then we can evaluate the integral using a primitive function $F(z) = \frac{2}{3}(z-1)^{\frac{3}{2}}$ (principal branch again). Then the integral is equal to

$$\begin{aligned} F(1-i) - F(1+i) &= \frac{2}{3}\Big((-i)^{\frac{3}{2}} - i^{\frac{3}{2}}\Big) \\ &= \frac{2}{3}\Big(e^{-\frac{1}{2}\pi i\cdot\frac{3}{2}} - e^{\frac{1}{2}\pi i\cdot\frac{3}{2}}\Big) \\ &= \frac{2}{3}\Big(e^{-\frac{3}{4}\pi i} - e^{\frac{3}{4}\pi i}\Big) \\ &= -\frac{4}{3} i \sin(\tfrac{3}{4}\pi) \\ &= -\frac{2}{3}\sqrt{2}\, i. \end{aligned}$$

b) $\displaystyle\int_{C_2(0)} \frac{\sin(z)}{4z^2+1}\,dz$, where $C_2(0)$ is positively oriented.
Using Cauchy's theorem for multiply connected regions we have

$$\int_{C_2(0)} \frac{\sin(z)}{4z^2+1}\,dz = \int_{C_{\frac{1}{4}}(i/2)} \frac{f_1(z)}{z-i/2}\,dz + \int_{C_{\frac{1}{4}}(-i/2)} \frac{f_2(z)}{z+i/2}\,dz,$$

with $f_1(z) = \frac{\sin(z)}{4(z+i/2)}$ and $f_2(z) = \frac{\sin(z)}{4(z-i/2)}$. f_1 is analytic on and inside $C_{\frac{1}{4}}(i/2)$ and f_2 is analytic on and inside $C_{\frac{1}{4}}(-i/2)$. Then using Cauchy's integral formula we find that the integral equals

$$\begin{aligned} 2\pi i\Big(f_1(i/2) + f_2(-i/2)\Big) &= 2\pi i\Big(\frac{\sin(i/2)}{4i} + \frac{\sin(-i/2)}{-4i}\Big) \\ &= \pi \sin(i/2) \\ &= -\frac{1}{2} i\pi(e^{-\frac{1}{2}} - e^{\frac{1}{2}}). \end{aligned}$$

6. Find all entire functions f with the property $|f'(z)| \geq 1$ for all $z \in \mathbb{C}$.
Hint: Liouville's theorem.

Consider the function $g = 1/f'$. Since f is entire, f' is also entire. And since $f'(z) \neq 0$ for all z, the function g is also entire. Then $|g(z)| \leq 1$, so g is bounded. By Liouville's theorem g is a constant. Then f' is also constant, $f' = a$ with $|a| \geq 1$. This implies $f(z) = az + b$ with $|a| \geq 1$ and $b \in \mathbb{C}$.

7. a) Let $m, n \in \mathbb{N}_0$ with $m \geq n$, and $r > 0$. Show that

$$\frac{1}{2\pi i}\int_{C_r(1)} \frac{z^m}{(z-1)^{n+1}}\,dz = \binom{m}{n},$$

where $C_r(1)$ has positive orientation.

Use Cauchy's generalized integral formula with $f(z) = z^m$, which is analytic on and inside $C_r(1)$. We have $f^{(n)}(1) = m(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$, so

$$\frac{1}{2\pi i}\int_{C_r(1)} \frac{f(z)}{(z-1)^{n+1}}\,dz = \frac{f^{(n)}(1)}{n!} = \frac{m!}{(m-n)!n!}.$$

b) Use part a to prove that

$$\binom{m}{n} \leq \frac{m^m n^{-n}}{(m-n)^{m-n}}.$$

For $m = n$ we have $\binom{m}{n} = 1$. In this case we have to interpret 0^0 as 1, and then the inequality holds (and is actually an identity).

Suppose $m > n$. From part a and the ML-inequality (or Cauchy's estimate) we find

$$\binom{m}{n} = \left|\frac{1}{2\pi i}\int_{C_r(1)} \frac{z^m}{(z-1)^{n+1}}\,dz\right| \leq \frac{1}{2\pi}\cdot 2\pi r \cdot \frac{M_r}{r^{n+1}} = \frac{M_r}{r^n},$$

where $M_r = \max\{|z^m| : |z-1| = r\}$. Using $|z^m| = |z|^m$ and $|z| \leq |z-1| + 1$ we find $M_r \leq (r+1)^m$, so that

$$\binom{m}{n} \leq \frac{(r+1)^m}{r^n} := F(r).$$

This holds for all $r > 0$, so we determine the minimum of F. The derivative to r is

$$\begin{aligned} F'(r) = \frac{d}{dr}\frac{(r+1)^m}{r^n} &= \frac{mr^n(r+1)^{m-1} - nr^{n-1}(r+1)^m}{r^{2n}} \\ &= \frac{(r+1)^{m-1}(mr - n(r+1))}{r^{n+1}}. \end{aligned}$$

This equals zero for $r = n/(m-n)$ and F' changes sign here (from + to $-$), so F has its minimum in $n/(m-n)$. We conclude that

$$\binom{m}{n} \leq \frac{(r+1)^m}{r^n} \leq F\left(\frac{n}{m-n}\right) = \frac{\left(\frac{m}{m-n}\right)^m}{\left(\frac{n}{m-n}\right)^n} = \frac{m^m n^{-n}}{(m-n)^{m-n}}.$$