Exam Discrete Optimization

22 January 2021, 13.30–16.45

The exam consists of 5 questions. In total you can obtain 90 points. The final grade is $1 + \#\text{points}/10$ rounded to the nearest integer or half-integer.

This is an open-book exam. It is NOT allowed to discuss with anyone else. If you have any questions regarding the exam, or technical questions regarding uploading of your answer, please contact David de Laat at d.delaat@tudelft.nl.

Please review the instructions posted on the announcement page for the course. The most important points are repeated below:

- Write your answers **by hand** and start each exercise on a new sheet.
- On your first answer sheet, you should write down the following statement: "I promise that I have not used unauthorized help from people or other sources for completing my exam. I created the submitted answers all by myself during the time slot that was allocated for that specific exam part."
- When scanning your work place your student-ID on the first page.
- Scan your work and submit it on brightspace as **one single pdf-file** no later than **16.45**.
- In case there is an announcement during the exam (for instance to correct a mistake in a question) this will be posted on brightspace and send by email using the classlist.

Good luck!

1. (a) (10 points) Figure 1 shows a graph G with a matching M indicated in red. Show how we can use the blossom algorithm to find a perfect matching starting from M. Show how this can be done with no blossom shrinking operations, and show how this can be done with at least one blossom shrinking operation. Show the intermediate steps.

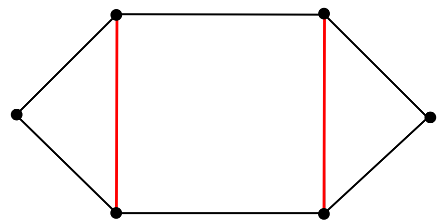

Figure 1

(b) (10 points) Figure 2 shows a bipartite graph $G = (V, E)$ with costs c_e on the edges. Let M be the matching consisting of the red edge. The labels on the nodes are a feasible solution y to the dual linear program

$$\begin{aligned} \text{maximize } & \sum_{v \in V} y_v \\ \text{subject to } & y_v + y_w \leq c_{vw} \text{ for all } vw \in E. \end{aligned}$$

Use the Hungarian algorithm (a.k.a. Hungarian method) to find a min-cost perfect matching starting from M and y. Give the optimal matching and optimal dual solution y you obtain.

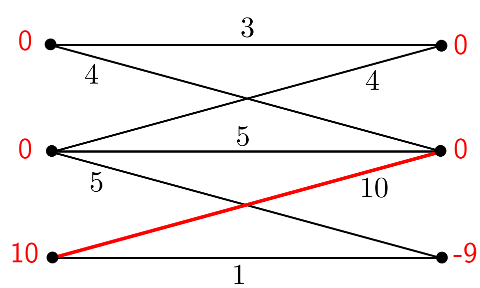

Figure 2

2. Consider the min-cost flow problem in the directed graph $G = (V, E)$ in Figure 3. The numbers on the nodes are the demands b_v, and the numbers on the arcs are the costs c_e, capacities u_e, and flows x_e (in that order).

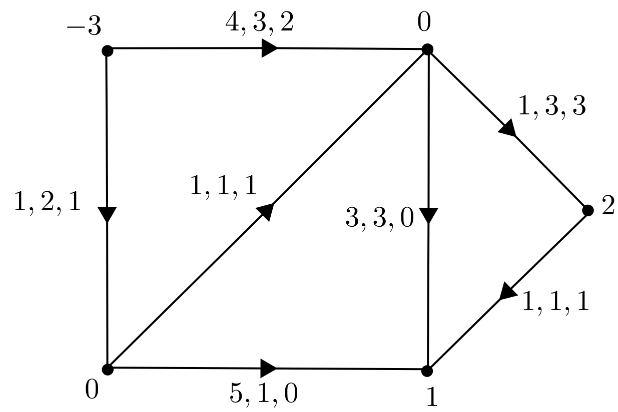

Figure 3

(a) (7 points) Give the auxiliary digraph $G(x)$.

(b) (10 points) Show there are no negative cost directed circuits in the auxiliary graph by solving a shortest path problem. Explain your answer.

(c) (6 points) The dual of the min-cost flow linear program is given by

$$\begin{aligned} \text{maximize } & \sum_{v \in V} b_v y_v - \sum_{vw \in E} u_{vw} z_{vw} \\ \text{subject to } & -y_v + y_w - z_{vw} \leq c_{vw} \text{ for all } vw \in E, \\ & z_e \geq 0 \text{ for all } e \in E. \end{aligned}$$

Give an optimal solution to this linear program for the given graph. Why does this prove optimality of the flow x as indicated in Figure 3?

3. Let $G = (V, E)$ be an undirected graph and consider the polytope P consisting of the vectors $x \in \mathbb{R}^E$ that satisfy

- $x_e \geq 0$ for all $e \in E$,
- $\sum_{e \in \delta(v)} x_e = 1$ for all $v \in V$.

(a) (5 points) Show that for each $U \subseteq V$ with $|U| \geq 3$ odd, the inequality

$$\sum_{e \in \gamma(U)} x_e \leq \frac{|U| - 1}{2}$$

is valid for the integral points in P.

(b) (5 points) Explain why the inequalities in (a) are Gomory-Chvátal cutting planes of P.

(c) (5 points) In the lecture we showed that the convex hull of the characteristic vectors of perfect matchings is the polytope consisting of the vectors $x \in \mathbb{R}^E$ satisfying

- $x_e \geq 0$ for all $e \in E$,
- $\sum_{e \in \delta(v)} x_e = 1$ for all $v \in V$,
- $\sum_{e \in \delta(U)} x_e \geq 1$ for all $U \subseteq V$, $|U| \geq 3$ odd.

Use this to show the convex hull of the characteristic vectors of perfect matchings is the set of vectors $x \in P$ that satisfy

$$\sum_{e \in \gamma(U)} x_e \leq \frac{|U| - 1}{2}$$

for all $U \subseteq V$ with $|U| \geq 3$ odd.

(d) (5 points) What is the Chvátal rank of P?

4. Let $G = (V, E)$ be a bipartite graph and consider the linear program

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_v \geq 0 \text{ for } v \in V, \\ & x_v + x_w \geq 1 \text{ for } vw \in E. \end{aligned}$$

(a) (10 points) Use total unimodularity to show the feasible set of this linear program is an integral polytope.

(b) (7 points) Write down the dual linear program. Is the linear system describing the feasible set of the above minimization problem totally dual integral?

5. (10 points) Let $G = (V, E)$ be a graph and $c \in \mathbb{R}^E$. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & c^\mathsf{T} x \\ \text{subject to} \quad & 0 \leq x_e \leq 1 \text{ for all } e \in E, \\ & \sum_{e \in T} x_e \geq 1 \text{ for all spanning trees } T \text{ in } G. \end{aligned}$$

Explain in a few sentences why this linear program can be solved efficiently (in polynomial time) in terms of the input size of G and c. You can use any of the results and algorithms from the lectures and the book.

End of test