Exam Markov Processes
AM2570
April 7, 2020, 13:30–16:30

- Every answer must be supplemented by adequate derivation, explanation and/or calculation, or it will receive no credit; write a mini-story in which you explain the steps that lead you to the answer. You may write yours answers in Dutch or English.
- Every part of every question has the same weight; there are 10 parts.
- On the first page of your solutions, leave some space and at the end of the exam copy and sign this:

  I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud.

- Please, also indicate somewhere near the beginning which of the following timeslots you are available for the face-to-face check, in the order of your preference: 10:00–12:00, 12:00–14:00, 14:00–16:00, all on Thursday, April 9th.
- Alarm set? Go and good luck!

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1. Given is a branching process X with $X_0 = 1$ and the size X_1 of the first generation has distribution

$$P(X_1 = 0) = \frac{1}{2}, \quad P(X_1 = 1) = p, \quad P(X_1 = 3) = \frac{1}{2} - p,$$

where $0 \le p \le \frac{1}{2}$. Note: the number of offspring is 0, 1, or 3 (three).

a. Determine for which values of p the extinction probability e equals 1.

b. For those values of p for which $e < 1$, determine the extinction probability e explicitly as a function of p. *Hint: the equation $G(s) - s = 0$ you need to solve always has a root $s = 1$, therefore is equivalent to $(s-1)H(s) = 0$ for some H.*

2. **a.** Let X_i, $i = 1, 2$ be independent random variables, X_i with $Exp(\rho_i)$-distribution. Verify (for yourself) that

$$P(t < X_1 < X_2) = \int_t^\infty P(x < X_2)\, f_{X_1}(x) \mathrm{d}x = \frac{\rho_1}{\rho_1 + \rho_2}\, \mathrm{e}^{-(\rho_1+\rho_2)\,t},$$

by conditioning on X_1. Define $Z = \min\{X_1, X_2\}$ and $A = \{X_1 < X_2\}$. Show that the random variable Z is independent of the event A; this may be done by showing that the events $\{Z > t\}$ and A are independent for all $t \ge 0$.

A device is subject to shocks, generated by a Poisson process at rate λ. The first two shocks do not yet interfere with its operation, but upon the third shock, with probability $\frac{1}{2}$ the device fails immediately, and with probability $\frac{1}{2}$ it enters a detiorated state from which it fails for certain when the next shock arrives. Upon failure, a repair robot is signalled immediately, but the robot needs an exponentially distributed time with parameter τ to reach the device; the repair time is exponential with rate μ. After that, the device is a good as new.

b. Explain why the state of the device may be modeled as a continuous-time Markov chain and draw a rate diagram.

(continued on the next page)

3. The transition matrix P of the Markov chain $\{X_n, n \geq 0\}$ with states $\{0, 1, 2, 3\}$ is given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

a. State the communicating classes, indicating which of them are closed and for each state indicate whether it is transient or recurrent.

b. Let H be the hitting time of the set $\{0, 1, 2\}$. Determine the distribution of X_H, if it is given that $X_0 = 3$, supporting your answer using the properties of the Markov chain.

c. Compute for $i = 0, 1, 2, 3$,

$$\lim_{n\to\infty} \mathrm{P}(X_n = i | X_0 = 3)\,.$$

4. In a barber shop with two barbers there are, next to the two barber chairs, two seats for waiting customers. Assume the arrival process of customers is Poisson at rate λ and that the barbers need an $Exp(\mu)$-distributed time to serve a customer; these service times are independent. Arriving customers that find all four seats occupied leave again and never return.

a. For the Markov chain that matches this description, describe the state space S and draw the rate diagram.

b. For $\lambda = 3$, $\mu = 2$, determine the stationary distribution π by the "rate in = rate out" principle.

c. For the same parameters determine the long term average number of customers waiting for their haircut.

Answers and partial solutions

1a $\mathrm{E}[X_1] = \mu = \frac{3}{2} - 2p$. By Theorem 9.22 (note the condition on p_1 is satisfied): $e = 1$ iff $\mu \le 1$, and this is equivalent to $\frac{1}{4} \le p \le \frac{1}{2}$.

1b From **1a** combined with the theorem: $e < 1$ iff $0 \le p < \frac{1}{4}$; we restrict ourselves to these p. Now,

$$G(s) = \mathrm{E}\left[s^{X_1}\right] = \tfrac{1}{2} + ps + (\tfrac{1}{2} - p)s^3$$

and

$$G(s) - s = \tfrac{1}{2}(s-1)[(1-2p)s^2 + (1-2p)s - 1].$$

So, the reduced equation is equivalent with:

$$s^2 + s - \frac{1}{1-2p} = 0 \quad \text{and} \quad (s + \tfrac{1}{2})^2 = \frac{1}{1-2p} + \frac{1}{4}.$$

Solution:

$$e = \frac{1}{2}\sqrt{\frac{5-2p}{1-2p}} - \frac{1}{2}.$$

2a Starting from

$$\mathrm{P}(t < X_1 < X_2) = \frac{\rho_1}{\rho_1 + \rho_2}\mathrm{e}^{-(\rho_1+\rho_2)t},$$

we see that $\mathrm{P}(X_1 < X_2) = \rho_1/(\rho_1 + \rho_2)$, by substituting $t = 0$ (though we already knew that). Immediate consequence: $\mathrm{P}(Z > t \,|\, A) = \mathrm{e}^{-(\rho_1+\rho_2)t}$, i.e., the conditional distribution of Z is $Exp(\rho_1 + \rho_2)$. However, we know this to be the *un*conditional distribution of Z, which proves the independence of A.

2b In the Poisson process the interarrival times are exponential with rate λ, so the transitions $0 \to 1 \to 2 \to$ happen at rate λ. From state 2 the process jumps to state F (failed) or to state D (deteriorated), each with probability 1/2, meaning that $r_{2D} = r_{2F} = \frac{1}{2}$. Since $q(2) = \lambda$, therefore $q_{2D} = q_{2F} = \lambda/2$. This is the only "complex transition," all the other transition are "deterministic," i.e., after an exponential sojourn time the chain moves to "the next state." See the diagram, that I haven't managed to include sofar.... The transition from state F to 0 goes via state R; the time this takes is the travel time.

3a There are two communicating classes: $\{0, 1\}$ (closed, recurrent); $\{2\}$ (closed, recurrent). State 3 is transient. (All states are aperiodic because $p_{ii} > 0$ for all i.)

3b Conditioning on $H = n$ we find for $j \in \{0, 1, 2\}$:

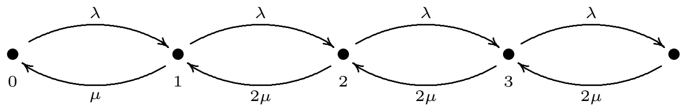
$$\begin{aligned}
\mathrm{P}(X_H = j \,|\, H = n, X_0 = 3) &= \mathrm{P}(X_n = j \,|\, X_0 = X_1 = \cdots = X_{n-1} = 3 \neq X_n) \\
&= \mathrm{P}(X_n = j \,|\, X_{n-1} = 3, X_n \neq 3) \\
&= \frac{\mathrm{P}(X_n = j, X_{n-1} = 3)}{\mathrm{P}(X_n \neq 3, X_{n-1} = 3)} \\
&= \frac{\mathrm{P}(X_n = j \,|\, X_{n-1} = 3)}{\mathrm{P}(X_n \neq 3 \,|\, X_{n-1} = 3)} = \frac{p_{3j}}{1 - p_{33}} = \frac{1}{3},
\end{aligned}$$

where in the first step the definition of H is used and in the second, the Markov property. Since the final answer does not depend on n, it equals $\mathrm{P}(X_H = j \,|\, X_0 = 3)$, demonstrating that X_H is uniformly distributed on $\{0, 1, 2\}$.

3c First compute the invariant measure of the Markov chain restricted to the two states $\{0, 1\}$; it is $\pi = (\frac{4}{7}, \frac{3}{7})$. From the previous part we know that the chain enters $\{0, 1, 2\}$ uniformly distributed over the states, whence

$$p_{30}^{(n)} \to \frac{2}{3}\frac{4}{7} = \frac{8}{21}, \quad p_{31}^{(n)} \to \frac{2}{3}\frac{3}{7} = \frac{2}{7}, \quad p_{32}^{(n)} \to \frac{1}{3},$$

where the factor 2/3 in the first two expressions is $\mathrm{P}(X_H \in \{0, 1\})$. Finally, $p_{33}^{(n)} \to 0$, because state 3 is transient.



Figuur 1: The transition intensities for the barbershop

4a As state we take the number of occupied seats (barber chairs and waiting chairs), so $S = \{0, 1, 2, 3, 4\}$. For the rate diagram see Figure 1.

4b "Rate in = rate out" yields the following equations:

$$\begin{aligned}
\lambda\pi_0 &= \mu\pi_1 \\
\lambda\pi_0 + 2\mu\pi_2 &= (\lambda + \mu)\pi_1 \\
\lambda\pi_1 + 2\mu\pi_3 &= (\lambda + 2\mu)\pi_2 \\
\lambda\pi_2 + 2\mu\pi_4 &= (\lambda + 2\mu)\pi_3 \\
\lambda\pi_3 &= 2\mu\pi_4.
\end{aligned}$$

Substracting equation $i - 1$ from equation i, $i = 1, \ldots, 4$, we obtain:

$$\lambda\pi_0 = \mu\pi_1 \quad \text{and} \lambda\pi_i = 2\mu\pi_{i+1} \quad i = 2, 3, 4.$$

Write $\rho = \lambda/(2\mu) = 3/4$, then all the probabilities can be expressed in terms of π_0:

$$\pi_i = 2\rho^i\pi_0 \quad i = 2, 3, 4.$$

From $\sum_{i=0}^{4} \pi_i = 1$ follows:

$$\pi_0 \left(1 + \sum_{i=1}^{4} \rho^i\right) = 1 \quad \text{or} \quad \pi_0 \left(2\frac{1-\rho^5}{1-\rho} - 1\right) = 1 \quad \text{i.e.} \quad \pi_0 = \frac{1-\rho}{1+\rho-2\rho^5}.$$

Numerical answers:

$$\pi = \frac{1}{653}(128, 192, 144, 108, 81) \approx (0.196, 0.294, 0.221, 0.165, 0.124).$$

4c In state 3 and 4, respectively 1 and 2 customers are waiting. The long term average of the number of customers waiting is the expected number under the stationary distribution, so $0 \cdot (\pi_0 + \pi_1 + \pi_2) + 1 \cdot \pi_3 + 2 \cdot \pi_4$. Numerical value: $270/653 \approx 0.413$.