Exam Markov Processes AM2570
April 6, 2021, 13:30–16:30

- Every answer must be supplemented by adequate derivation, explanation and/or calculation, or it will receive no credit; write a mini-story in which you explain the steps that lead you to the answer. You may write yours answers in Dutch or English.
- Every part of every question has the same weight; there are 10 parts.
- On the first page of your solutions, leave some space and at the end of the exam copy and sign this:

  I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud.

- Please, also indicate somewhere near the beginning which of the following timeslots you are available for the face-to-face check, in the order of your preference: 10:00–12:00, 14:00–16:00, on Tuesday April 13th. If we want to speak with you, you will receive an invitation before Monday, April 12th, 18:00.
- Alarm set? Go and good luck!

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1. Suppose that the branching process $(X_n)_{n\geq 0}$ describes the number of radioactive particles in a controlled nuclear reaction. So $X_n = k$ states there are k radioactive particles present at time n. The probability distribution of the number of "offspring" Z of a single particle is for $k \geq 0$ given by:

$$\mathrm{P}(Z = k) = (1-p)p^k,$$

where p is a parameter satisfying $0 < p < 1$. Here $Z = k$ means that this particle has k offspring. The particle itself is then lost as a radioactive particle in the next generation (so it 'dies' after having had offspring in the next generation). The number of offspring of different particles is independent.

a) Determine the expected size $\mu = \mathrm{E}[Z]$ of X_1 if it is given that $X_0 = 1$.

b) Determine in two ways the values of $p \in (0,1)$ for which the probability e of ultimate extinction of this branching process equals 1.

2. Let $(X_n)_{n\geq 0}$ be a discrete Markov chain with finite state space $S = \{0, 1, \ldots, m\}$, and with P_X as its matrix of transition probabilities. Define for $n = 0, 1, 2, \ldots$ the random variable Y_n by $Y_n = X_{3n}$.

a) Show that $(Y_n)_{n\geq 0}$ is also a discrete Markov chain with state space S; i.e. show that the *Markov property* holds: i.e. that the (conditional) probability

$$\mathrm{P}(Y_{n+1} = j \,|\, Y_n = i, Y_{n-1} = i_{n-1}, Y_{n-2} = i_{n-2}, \ldots, Y_1 = i_1, Y_0 = i_0)$$

(for those $i_0, i_1, \ldots, i_{n-1}, i, j \in S$ with $\mathrm{P}(Y_n = i, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0) > 0$) is equal to the (conditional) probability

$$\mathrm{P}(Y_{n+1} = j \,|\, Y_n = i).$$

What is the matrix P_Y of transition probabilities of the Markov chain $(Y_n)_{n\geq 0}$?

b) Show that if the Markov chain $(Y_n)_{n\geq 0}$ is irreducible, also the Markov chain $(X_n)_{n\geq 0}$ is irreducible. Does the converse hold? (so if $(X_n)_{n\geq 0}$ is irreducible, is also $(Y_n)_{n\geq 0}$ irreducible?). Give a proof or a counter-example.

3. Let $(X_n)_{n\geq 0}$ be a discrete Markov chain with state space $S = \{1, 2, 3, 4, 5\}$, and with matrix of transition probabilities P, given by

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

a) Identify the communicating classes of P. Which classes are closed? Which states are recurrent, and which are transient? Are there absorbing states?

b) Show that P has infinitely many different invariant distributions

$$\pi = (\, \pi_1 \ \pi_2 \ \cdots \ \pi_5 \,).$$

4. In this exercise we consider a variant of Ehrenfest's gas-model from Chapter 3 of the Reader.[1]

Let N be a positive integer. We distribute N black balls and N white balls over two vases, one of which is black, and the other vase is white. Each vase gets exactly N balls. At time n (with $n = 0, 1, 2, 3, \ldots$) we simultaneously and randomly draw one ball from each vase, and put each ball in the vase where the other ball was drawn from. Set

$X_n =$ the number of black balls in the black vase at time n immediately after the exchange.

Then $(X_n)_{n\geq 0}$ is a discrete Markov chain on the state space $S = \{0, 1, \ldots, N\}$.

a) Determine the matrix P_2 of transition probabilities for $N = 2$. Determine the unique distribution $\pi = (\, \pi_0 \ \pi_1 \ \pi_2 \,)$ for $N = 2$ (indicate why π is unique).

b) Determine P_N for a general N (where $N \geq 2$). Someone claims that for $N \geq 2$ we have for the invariant distribution $\pi = (\, \pi_0 \ \pi_1 \ \cdots \ \pi_N \,)$ that:

$$\pi_k = \binom{N}{k}^2 \cdot \pi_0, \quad \text{voor } k = 0, 1, \ldots, N,$$

and

$$\pi_0 = \frac{1}{\binom{2N}{N}}.$$

Setting $\pi P_N = (\, x_0 \ x_1 \ x_2 \ \cdots \ x_N \,)$, check whether $x_0 = \pi_0$, $x_1 = \pi_1$ and $x_2 = \pi_2$.

5. Consider the taxi-stand at *Delft International Airport*, which is a small airport, where day and night taxis arrive according to a Poisson process with "rate" of 1 taxi per minute (so $\lambda = 1$), and where customers arrive according to a Poisson-process with "rate" 2 customers per minute (so $\mu = 2$). These rates are independent of the number of taxis and customers waiting, and the two Poisson-processes are also independent. A taxi waiting at the taxi-stand will always wait for a customer (even if there are another 100 taxis waiting at the

[1] You don't need to know this Ehrenfest gas-model discussed in the reader to answer this exercise.

taxi-stand). However, passengers arriving at an empty taxi-stand immediately leave (and take a bus or other means of transportation), so these customers are lost for the taxi's. The taxi-central wonders how many taxis are lost and asks you–as a mathematician– to analyze the situation.

a) What is the expected number of taxis waiting at the taxi-stand in 'steady-state'? What is the fraction of arriving passengers lost for transportation by taxi?

b) After hearing your answers to part a), the taxi-central decides to buy a number of local taxi-companies in order to raise the arrival-rate of the taxis to $\lambda = 2$ taxis per minute. Is this a good idea? Why (not)?

The End