Take home Exam Martingales, Brownian motion and stochastic calculus (WI4430). January 27th, 9:00-12:00. VERSION 1

a) The exam has a theory part consisting of ten true/false questions. This part tests your **understanding** of the definitions and of the theorems. The theory part is on 20 points (2 points for each question). The exercise part consists of 10 questions each on 2 points.

b) The second reader of the exam is Drs. Simone Floreani.

c) The take home exam will be available on Brightspace, January 27th at 8:30. You are asked to send the scanned solutions before January 27th, 12:30 (13:00 for the people with extra time). Deliver your scanned solution as a UNIQUE pdf file (multiple files will not be accepted) and name it Surname-Name-exam.pdf. Send it to the following two email addresses: f.h.j.redig@tudelft.nl and fhjredig@gmail.com.

d) The exam is open book, i.e., you are allowed to use book and course notes. Notice that you are not allowed to do internet searches during the exam.

Please start your exam by writing down the code of honour sentence:
"I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."
An exam without this sentence will not be corrected.

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1. Are the following statements true or false? Justify (concisely) your answer (with an unjustified true or false answer, you do not earn points).

   a) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G}' \subset \mathscr{G} \subset \mathscr{F}$ are two sub-σ-algebras of $\mathscr{F}$. Then we have, for $X \in L^2(\Omega, \mathscr{G}', \mathbb{P}), Y \in L^2(\Omega, \mathscr{G}, \mathbb{P})$
   $$\mathbb{E}(XY|\mathscr{G}) = XY.$$

b) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G} \subset \mathscr{F}$ is a sub-σ-algebra of $\mathscr{F}$. If X and Y are independent bounded random variables, and B is an event which is independent of $\mathscr{G}$ then we have

$$\mathbb{E}(XY1_B|\mathscr{G}) = \mathbb{P}(B)\mathbb{E}(X|\mathscr{G})\mathbb{E}(Y|\mathscr{G}).$$

c) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. Moreover, $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is another filtration, and we have for all $n \in \mathbb{N}$, $\mathscr{F}_n \supset \mathscr{G}_n$. Then every martingale w.r.t. the filtration $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is also a martingale w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

d) The exponential of a supermartingale is a submartingale.

e) $\{Y_i, i \in \mathbb{N}\}$ is a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \ldots, Y_n)$ is the associated natural filtration. Then

$$\tau = \inf\left\{k : \sum_{i=1}^{k} Y_i > 10\right\}$$

is a bounded stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

f) An L^2-bounded martingale is a Doob martingale.

g) An L^3-bounded martingale converges almost surely.

h) Let $\alpha > 0$, then we define

$$D_\alpha^+ W(t) := \limsup_{s \to t} \frac{|W(s) - W(t)|}{|t-s|^\alpha}$$

Then for all $\alpha > 1/2$, and for all $t > 0$, $D_\alpha^+ W(t) = \infty$ almost surely.

i) Let $\{W(t), t \geq 0\}$ denote Brownian motion. Define

$$Z(t) = \sqrt{t}(W(1))$$

then $\{Z(t) : t \geq 0\}$ is also a Brownian motion.

j) Let (X_1, X_2) be a Gaussian vector. Assume that

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = 0.$$

Then the vector (Y_1, Y_2) given by

$$\begin{aligned} Y_1 &= \frac{X_1 + X_2}{\sqrt{2}} \\ Y_2 &= \frac{X_1 - X_2}{\sqrt{2}} \end{aligned}$$

is a Gaussian vector and has the same distribution as (X_1, X_2).

2. Let $\{Y_i, i \in \mathbb{N}\}$ denote an i.i.d. sequence of standard normal random variables.

   Additionally, we have a random variable X which is independent of the sequence $\{Y_i, i \in \mathbb{N}\}$ and which has a Bernoulli distribution, i.e., X takes values in $\{0, 1\}$ with $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0)$, where $0 < p < 1$.

   We denote by $\mathscr{F}_n = \sigma\{Y_i, 1 \leq i \leq n\}$ the natural filtration associated to the sequence $\{Y_i, i \in \mathbb{N}\}$. Moreover we denote $S_n = \sum_{i=1}^n Y_i$ for $n \geq 1$, and $S_0 = 0$.

   You are allowed to use that $\mathbb{E}e^{tY_i} = e^{\frac{t^2}{2}}$ for all $t \in \mathbb{R}$, and $\mathbb{E}(Y_i^4) = 3$.

   a) Compute $\mathbb{E}(Y_2^2|X)$.

   b) Compute $\mathbb{E}(e^{tXS_n}|\mathscr{F}_m)$ for $t \in \mathbb{R}$, $1 \leq m \leq n$.

   c) Use the Kolmogorov-Doob inequality to prove that for all $\delta > 0$

   $$\lim_{n\to\infty} \sqrt{\frac{1}{n^{1+\delta}}} (\max_{1\leq k\leq n} S_k) = 0$$

   where the convergence is in probability.

   d) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Show that

   $$X_n = e^{\sum_{i=1}^n (a_i Y_i - \frac{1}{2} a_i^2)} \tag{1}$$

   is a martingale w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

   e) Show the martingale in item d) converges in L^2 and almost surely whenever $\sum_{i=1}^\infty a_i^2 < \infty$.

3. Let $\{W(t) : t \geq 0\}$ denote Brownian motion, and let $\{\mathscr{F}_t, t \geq 0\}$ denote its natural filtration. Denote, for $a, b \in (0, \infty)$:

   $$\tau_{a,b} = \inf\{t \geq 0 : W(t) \in \{-a, b\}\}, \tag{2}$$

   and

   $$T_a = \inf\{t \geq 0 : W(t) = a\}. \tag{3}$$

   You are allowed to use that $T_a, \tau_{a,b}$ are finite stopping times. When you are asked to use an appropriate martingale to compute the expectation of (a function of a) stopping time, you do not have to prove the martingale property, but you do have to argue an exchange of limits and expectations whenever necessary.

a) Compute $\mathbb{E}(W(t)W(s)^2|\mathscr{F}_r)$ for $0 < r < s < t$.

b) Compute $\mathbb{E}(\sup_{0\le t\le 1}(W(s+t) - W(s))|\mathscr{F}_s)$, for $s > 0$.

c) Use an appropriate martingale to compute $\mathbb{E}(\tau_{a,b})$.

d) Use an appropriate martingale to compute $\mathbb{E}e^{-\lambda T_a}$ for $\lambda > 0$.

e) Prove that for $a > 0$, with probability one, Brownian motion contains a zero in the interval $(0, a)$, i.e., there exists $0 < t < a$ such that $W(t) = a$.