Take home Exam Martingales, Brownian motion and stochastic calculus (WI4430). January 27th, 9:00-12:00. VERSION 2

a) The exam has a theory part consisting of ten true/false questions. This part tests your **understanding** of the definitions and of the theorems. The theory part is on 20 points (2 points for each question). The exercise part consists of 10 questions each on 2 points.

b) The second reader of the exam is Drs. Simone Floreani.

c) The take home exam will be available on Brightspace, January 27th at 8:30. You are asked to send the scanned solutions before January 27th, 12:30 (13:00 for the people with extra time). Deliver your scanned solution as a UNIQUE pdf file (multiple files will not be accepted) and name it Surname-Name-exam.pdf. Send it to the following two email addresses: f.h.j.redig@tudelft.nl and fhjredig@gmail.com.

d) The exam is open book, i.e., you are allowed to use book and course notes. Notice that you are not allowed to do internet searches during the exam.

Please start your exam by writing down the code of honour sentence:
"I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."
An exam without this sentence will not be corrected.

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1. Are the following statements true or false? Justify (concisely) your answer (with an unjustified true or false answer, you do not earn points).

   a) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G}' \subset \mathscr{G} \subset \mathscr{F}$ are two sub-σ-algebra of $\mathscr{F}$. If $B \in \mathscr{G}'$, then for every $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$

$$\mathbb{E}(1_B X|\mathscr{G}) = 1_B \mathbb{E}(X|\mathscr{G}')$$

b) Let $\{Y_i, i \in \mathbb{N}\}$ denote a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \ldots, Y_n)$ is the associated natural filtration. Then we have

$$\mathbb{E}(e^{Y_2 - Y_1} | \mathscr{F}_1) = e^{-Y_1 + \frac{1}{2}}$$

c) The square root of a non-negative martingale is a supermartingale.

d) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. If $\{X_n, n \in \mathbb{N}\}$ is a martingale w.r.t. $\{\mathscr{F}_n, n \in \mathbb{N}\}$, then $\{X_{2n}, n \in \mathbb{N}\}$ is a martingale w.r.t $\{\mathscr{F}_{2n}, n \in \mathbb{N}\}$.

e) A random time is a stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$ if and only if the event $\{\tau \geq n\}$ belongs to $\mathscr{F}_n$ for all $n \in \mathbb{N}$.

f) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. Moreover, $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is another filtration, and we have for all $n \in \mathbb{N}$ $\mathscr{F}_n \supset \mathscr{G}_n$. Then every martingale w.r.t. the filtration $\{\mathscr{G}_n, n \in \mathbb{N}\}$ martingale is also a martingale w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

g) $\{Y_i, i \in \mathbb{N}\}$ is a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \ldots, Y_n)$ is the associated natural filtration. τ is a bounded stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$. Then

$$\mathbb{E}\left(\left(\prod_{i=1}^{\tau} e^{Y_i}\right) e^{-\frac{\tau}{2}}\right) = 1$$

h) If a non-negative martingale $\{X_n, n \in \mathbb{N}\}$ converges in L^1, and is bounded from below by a strictly positive constant, i.e., $X_n \geq a > 0$ for some $a > 0$, and for all $n \in \mathbb{N}$, then the sequence $\{1/X_n, n \in \mathbb{N}\}$ converges almost surely and in L^1.

i) Let $\{W(t), t \geq 0\}$ denote Brownian motion, then, for $0 < s < t$, the random variables $W(s)$ and $W(t)$ are independent.

j) Let $\{W(t), t \geq 0\}$ denote Brownian motion, then the process $\{W(t) - tW(1) + t^2 W(\frac{1}{2}) : t \geq 0\}$ is a Gaussian process.

2. Let $\{Y_i, i \in \mathbb{N}\}$ denote an i.i.d. sequence of standard normal random variables.

Additionally, we have a random variable X which is independent of the sequence $\{Y_i, i \in \mathbb{N}\}$ and which has the Bernoulli distribution, i.e., X takes values in $\{0, 1\}$ with $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0)$, where $0 < p < 1$.

We denote by $\mathscr{F}_n = \sigma\{Y_i, 1 \leq i \leq n\}$ the natural filtration associated to the sequence $\{Y_i, i \in \mathbb{N}\}$. Moreover we denote $S_n = \sum_{i=1}^n Y_i$ for $n \geq 1$, and $S_0 = 0$. You are allowed to use that $\mathbb{E}e^{tY_i} = e^{\frac{t^2}{2}}$ for all $t \in \mathbb{R}$, and $\mathbb{E}(Y_i^4) = 3$.

a) Compute $\mathbb{E}(S_2^X | \mathscr{F}_1)$.

b) Compute $\mathbb{E}(S_n^2 | \mathscr{F}_m)$ for $1 \leq m \leq n$.

c) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Determine the Doob-Meyer decomposition of the submartingale $Z_n := (\sum_{i=1}^n a_i Y_i)^2$. (You do not have to show that Z_n is a submartingale.)

d) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Show that if $\sum_{i=1}^\infty a_i^2 < \infty$, then
$$X_n = \sum_{i=1}^n a_i Y_i$$
is a martingale which converges almost surely and in L^2.

e) Denote for $a, b \in \mathbb{N}, a, b \geq 1$: $\tau_{a,b} = \inf\{n \in \mathbb{N} : S_n \in \{-a, b\}\}$. Show that $\{S_{n \wedge \tau_{a,b}}, n \in \mathbb{N}\}$ is a Doob martingale. If needed, you are allowed to use that $\mathbb{E}(\tau_{a,b}) < \infty$.

3. Let $\{W(t) : t \geq 0\}$ denote Brownian motion, $\{\mathscr{F}_t, t \geq 0\}$ its natural filtration. Denote, for $a, b \in (0, \infty)$:
$$\tau_{a,b} = \inf\{t \geq 0 : W(t) \in \{-a, b\}\}, \tag{1}$$
and
$$T_a = \inf\{t \geq 0 : W(t) = a\} \tag{2}$$
and finally
$$\mathscr{T}_{a,b} = \inf\{t > \tau_{a,b} : W(t) = 0\} \tag{3}$$
You are allowed to use without further justification that $T_a, \mathscr{T}_{a,b}, \tau_{a,b}$ are finite stopping times. When you are asked to use an appropriate martingale to compute the expectation of (a function of a) stopping time, you do not have to prove the martingale property, but you do have to argue an exchange of limits and expectations whenever necessary.

a) Use an appropriate martingale to compute $\mathbb{E}(e^{-\lambda T_a})$ for $\lambda > 0$.

b) Put $M(t) = \sup_{0 \leq s \leq t} W(s)$. Compute $\mathbb{E}[(M(t) - W(t))^4]$. You are allowed to use that for a standard normal random variable Z, $\mathbb{E}(Z^4) = 3$.

c) Show that for all $t > 0$:
$\mathbb{E}[(W(t \wedge \mathscr{T}_{a,b}) - W(t \wedge \tau_{a,b}))^2] = \mathbb{E}(t \wedge \mathscr{T}_{a,b} - t \wedge \tau_{a,b})$.

d) Show that with probability one for every $a > 0$, Browian motion has a zero in $(0, a)$, i.e., there exists $0 < t < a$ such that $W(t) = 0$.

e) Let $\{W_b(t), 0 \le t \le 1\}$ denote the Brownian bridge, i.e., $W_b(t) = W(t) - tW(1)$ for $0 \le t \le 1$. Compute the covariance between $W_b(t)$ and $W(t)$, for $0 \le t \le 1$.