# Take home Exam Martingales, Brownian motion and stochastic calculus (WI4430). January 27th, 9:00-12:00. VERSION 3

a) The exam has a theory part consisting of ten true/false questions. This part tests your **understanding** of the definitions and of the theorems. The theory part is on 20 points (2 points for each question). The exercise part consists of 10 questions each on 2 points.

b) The second reader of the exam is Drs. Simone Floreani.

c) The take home exam will be available on Brightspace, January 27th at 8:30. You are asked to send the scanned solutions before January 27th, 12:30 (13:00 for the people with extra time). Deliver your scanned solution as a UNIQUE pdf file (multiple files will not be accepted) and name it Surname-Name-exam.pdf. Send it to the following two email addresses: f.h.j.redig@tudelft.nl and fhjredig@gmail.com.

d) The exam is open book, i.e., you are allowed to use book and course notes. Notice that you are not allowed to do internet searches during the exam.

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**Please start your exam by writing down the code of honour sentence:**
*"I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."*
**An exam without this sentence will not be corrected.**

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1. Are the following statements true or false? Justify (concisely) your answer (with an unjustified true or false answer, you do not earn points).

   a) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G}' \subset \mathscr{G} \subset \mathscr{F}$ are two sub-$\sigma$-algebra of $\mathscr{F}$. If $B \in \mathscr{G}'$, then for every $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$

$$\mathbb{E}(1_B X|\mathscr{G}) = 1_B \mathbb{E}(X|\mathscr{G}')$$

b) Let $\{Y_i, i \in \mathbb{N}\}$ denote a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \dots, Y_n)$ is the associated natural filtration. Then we have

$$\mathbb{E}(e^{Y_2 - Y_1} | \mathscr{F}_1) = e^{-Y_1 + \frac{1}{2}}$$

c) The logarithm of a non-negative martingale is a supermartingale.

d) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. If $\{X_n, n \in \mathbb{N}\}$ is a martingale w.r.t. $\{\mathscr{F}_n, n \in \mathbb{N}\}$, then $\{X_{2n}, n \in \mathbb{N}\}$ is a martingale w.r.t $\{\mathscr{F}_{2n}, n \in \mathbb{N}\}$.

e) A random time is a stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$ if and only if the event $\{\tau \geq n\}$ belongs to $\mathscr{F}_n$ for all $n \in \mathbb{N}$.

f) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. Moreover, $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is another filtration, and we have for all $n \in \mathbb{N}$ $\mathscr{F}_n \supset \mathscr{G}_n$. Then every martingale w.r.t. the filtration $\{\mathscr{G}_n, n \in \mathbb{N}\}$ martingale is also a martingale w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

g) $\{Y_i, i \in \mathbb{N}\}$ denotes a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \dots, Y_n)$ is the associated natural filtration. $\tau$ is a bounded stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$. Then

$$\mathbb{E}\left(\prod_{i=1}^{\tau} e^{Y_i - \frac{1}{2}}\right) = 1$$

h) Let $\{X_n, n \in \mathbb{N}\}$ be a non-negative martingale which converges in $L^1$, and is bounded from below by a strictly positive constant, i.e., there exists $a > 0$ such that $X_n \geq a$ for all $n$. Then the sequence $\{\frac{1}{X_n}, n \in \mathbb{N}\}$ converges almost surely and in $L^1$.

i) Let $\{W(t), t \geq 0\}$ denote Brownian motion, then the random variables $W(t)$ and $W(s)$, with $t > s$, are independent.

j) Let $\{W(t), t \geq 0\}$ denote Brownian motion, then the process $\{W(t) - tW(1) + t^2 W(\frac{1}{2}) : t \geq 0\}$ is a Gaussian process.

2. Let $\{Y_i, i \in \mathbb{N}\}$ denote an i.i.d. sequence of random variables taking the values $\pm 1$ with equal probability, i.e., $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$. We denote by $\mathscr{F}_n = \sigma\{Y_i, 1 \leq i \leq n\}$ the natural filtration associated to the sequence $\{Y_i, i \in \mathbb{N}\}$. Moreover we denote $S_n = \sum_{i=1}^n Y_i$ for $n \geq 1$, and $S_0 = 0$. Denote, for $a \in \mathbb{N}, a \geq 2$:

$$\tau_{-a,a} = \inf\{n \in \mathbb{N} : S_n \in \{-a, a\}\}. \tag{1}$$

When you are asked to use an appropriate martingale to compute the expectation of (a function of a) stopping time, you do not have to prove the martingale property, but you do have to argue an exchange of limits and expectations whenever necessary.

a) Compute $\mathbb{E}(e^{XS_n}|\mathscr{F}_m)$ for $1 \leq m \leq n$.

b) Compute $\mathbb{E}(S_n^2|S_m)$ for $1 \leq m \leq n$.

c) Use an appropriate martingale to compute the expectation $\mathbb{E}(e^{-\lambda\tau_{-a,a}})$ for $\lambda > 0$.

d) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Show that if $\sum_i a_i^2 < \infty$, then

$$X_n = \sum_{i=1}^{n} a_i Y_i$$

is a martingale which converges almost surely and in $L^2$.

e) Same setting as in item d). Prove, for $\epsilon > 0$, the following upper bound for the probability

$$\mathbb{P}\left(\sup_{1\leq k\leq n} |X_k| > \epsilon\right) \leq \frac{1}{\epsilon^2}\sum_{i=1}^{n} a_i^2$$

3. Let $\{W(t) : t \geq 0\}$ denote Brownian motion, $\{\mathscr{F}_t, t \geq 0\}$ its natural filtration. Denote, for $a \in (0, \infty)$:

$$T_a = \inf\{t \geq 0 : W(t) = a\} \tag{2}$$

You are allowed to use without further justification that $T_a$ is a finite stopping time. When you are asked to use an appropriate martingale to compute the expectation of (a function of a) stopping time, you do not have to prove the martingale property, but you do have to argue an exchange of limits and expectations whenever necessary.

a) Compute $\mathbb{E}(W(t)e^{\lambda W(t)-\frac{1}{2}\lambda^2 t})$ for $\lambda \in \mathbb{R}$, $t > 0$.

b) Compute $\mathbb{E}(W(s)^4|W(t))$ for $0 < s < t$. You are allowed to use that for a standard normal random variable $Z$, $\mathbb{E}(Z^4) = 3$.

c) Show that $\mathbb{E}(W^2(t \wedge T_a)) = \mathbb{E}(t \wedge T_a)$ for all $t > 0$. Can you take the limit $t \to \infty$ to conclude that $\mathbb{E}(T_a) = a^2$?

d) Show that for $a < b$, $T_b - T_a$ and $T_{b-a}$ have the same distribution.

e) Let $\{W_b(t), 0 \leq t \leq 1\}$ denote Brownian bridge, i.e., $W_b(t) = W(t) - tW(1)$. Compute the covariance between $W_b(t)$ and $W(t)$, for $0 \leq t \leq 1$.