Take home Exam Martingales, Brownian motion and stochastic calculus (WI4430). January 27th, 9:00-12:00. VERSION 4

a) The exam has a theory part consisting of ten true/false questions. This part tests your **understanding** of the definitions and of the theorems. The theory part is on 20 points (2 points for each question). The exercise part consists of 10 questions each on 2 points.

b) The second reader of the exam is Drs. Simone Floreani.

c) The take home exam will be available on Brightspace, January 27th at 8:30. You are asked to send the scanned solutions before January 27th, 12:30 (13:00 for the people with extra time). Deliver your scanned solution as a UNIQUE pdf file (multiple files will not be accepted) and name it Surname-Name-exam.pdf. Send it to the following two email addresses: f.h.j.redig@tudelft.nl and fhjredig@gmail.com.

d) The exam is open book, i.e., you are allowed to use book and course notes. Notice that you are not allowed to do internet searches during the exam.

Please start your exam by writing down the code of honour sentence:
"I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."
An exam without this sentence will not be corrected.

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1. Are the following statements true or false? Justify (concisely) your answer (with an unjustified true or false answer, you do not earn points).

   a) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G}' \subset \mathscr{G} \subset \mathscr{F}$ are two sub-σ-algebras of $\mathscr{F}$. Then we have, for $X \in L^2(\Omega, \mathscr{G}', \mathbb{P}), Y \in L^2(\Omega, \mathscr{G}, \mathbb{P})$
   $$\mathbb{E}(XY|\mathscr{G}) = XY$$

b) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G} \subset \mathscr{F}$ is a sub-σ-algebra of $\mathscr{F}$. If X and Y are independent bounded random variables, and B is an event which is independent of $\mathscr{G}$, then we have

$$\mathbb{E}(XY1_B|\mathscr{G}) = \mathbb{P}(B)\mathbb{E}(X|\mathscr{G})\mathbb{E}(Y|\mathscr{G})$$

c) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. Additionally, $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is another filtration, and we have for all $n \in \mathbb{N}$ $\mathscr{F}_n \supset \mathscr{G}_n$. Then every martingale w.r.t. the filtration $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is also a martingale w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

d) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. If $\{X_n, n \in \mathbb{N}\}$ is a martingale w.r.t. $\{\mathscr{F}_n, n \in \mathbb{N}\}$, then $\{X_{2n}, n \in \mathbb{N}\}$ is a martingale w.r.t $\{\mathscr{F}_{2n}, n \in \mathbb{N}\}$.

e) Let $\{Y_i, i \in \mathbb{N}\}$ denote a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \ldots, Y_n)$ is the associated natural filtration. Then

$$\tau = \inf \left\{ k : \sum_{i=1}^{k} Y_i > 10 \right\}$$

is a bounded stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

f) An L^2-bounded martingale is a Doob martingale.

g) An L^3-bounded martingale converges almost surely and in L^1.

h) Let $\{W(t), t \geq 0\}$ denote Brownian motion. Then the process $X(t) = e^{-t}W(e^{2t})$ is a Gaussian process which is standard normally distributed at every $t > 0$.

i) Let $\{W(t), t \geq 0\}$ denote Brownian motion. Define

$$Z(t) = \sqrt{t}W(1)$$

then $\{Z(t) : t \geq 0\}$ is also a Brownian motion.

j) Let (X_1, X_2) be a Gaussian vector. Assume that

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = 0.$$

Then the vector (Y_1, Y_2) given by

$$Y_1 = \frac{X_1 + X_2}{\sqrt{2}}$$
$$Y_2 = \frac{X_1 - X_2}{\sqrt{2}}$$

is a Gaussian vector and has the same distribution as (X_1, X_2).

2. Let $\{Y_i, i \in \mathbb{N}\}$ denote an i.i.d. sequence of random variables taking the values ± 1 with equal probability, i.e., $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$. We denote by $\mathscr{F}_n = \sigma\{Y_i, 1 \le i \le n\}$ the natural filtration associated to the sequence $\{Y_i, i \in \mathbb{N}\}$. Moreover we denote $S_n = \sum_{i=1}^n Y_i$ for $n \ge 1$, and $S_0 = 0$. Denote, for $a, b \in \mathbb{N}, a, b \ge 2$:

$$\tau_{-a,a} = \inf\{n \in \mathbb{N} : S_n \in \{-a, a\}\}. \tag{1}$$

When you are asked to use an appropriate martingale to compute the expectation of (a function of a) stopping time, you do not have to prove the martingale property, but you do have to argue an exchange of limits and expectations whenever necessary.

a) Compute $\mathbb{E}(e^{XS_n}|\mathscr{F}_m)$ for $1 \le m \le n$.

b) Compute $\mathbb{E}(S_n^2|S_m)$ for $1 \le m \le n$.

c) Use an appropriate martingale to compute the expectation $\mathbb{E}(e^{-\lambda\tau_{-a,a}})$ for $\lambda > 0$.

d) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Show that if $\sum_i a_i^2 < \infty$, then

$$X_n = \sum_{i=1}^n a_i Y_i$$

is a martingale which converges almost surely and in L^2.

e) Same setting as in item d). Prove, for $\epsilon > 0$, the following upper bound for the probability

$$\mathbb{P}\left(\sup_{1\le k\le n} |X_k| > \epsilon\right) \le \frac{1}{\epsilon^2}\sum_{i=1}^n a_i^2$$

3. Let $\{W(t) : t \ge 0\}$ denote Brownian motion, and $\{\mathscr{F}_t, t \ge 0\}$ its natural filtration. Denote, for $a \in (0, \infty)$:

$$T_a = \inf\{t \ge 0 : W(t) = a\} \tag{2}$$

You are allowed to use without further justification that T_a is a finite stopping time. When you are asked to use an appropriate martingale to compute the expectation of (a function of a) stopping time, you do not have to prove the martingale property, but you do have to argue an exchange of limits and expectations whenever necessary.

a) Compute $\mathbb{E}(W(t)e^{\lambda W(t) - \frac{1}{2}\lambda^2 t})$ for $\lambda \in \mathbb{R}$, $t > 0$.

b) Compute $\mathbb{E}(W(s)^4 | W(t))$ for $0 < s < t$. You are allowed to use that for a standard normal random variable Z, $\mathbb{E}(Z^4) = 3$.

c) Show that $\mathbb{E}(W^2(t \wedge T_a)) = \mathbb{E}(t \wedge T_a)$ for all $t > 0$. Are you allowed to take the limit $t \to \infty$ to conclude that $\mathbb{E}(T_a) = a^2$?

d) Show that for $a < b$, $T_b - T_a$ and T_{b-a} have the same distribution.

e) Let $\{W_b(t), 0 \leq t \leq 1\}$ denote Brownian bridge, i.e., for $0 \leq t \leq 1$, $W_b(t) = W(t) - tW(1)$. Compute the covariance between $W_b(t)$ and $W(t)$, for $0 \leq t \leq 1$.