Take home Exam Martingales, Brownian motion and stochastic calculus (WI4430). January 27th, 9:00-12:00. VERSION 5

a) The exam has a theory part consisting of ten true/false questions. This part tests your **understanding** of the definitions and of the theorems. The theory part is on 20 points (2 points for each question). The exercise part consists of 10 questions each on 2 points.

b) The second reader of the exam is Drs. Simone Floreani.

c) The take home exam will be available on Brightspace, January 27th at 8:30. You are asked to send the scanned solutions before January 27th, 12:30 (13:00 for the people with extra time). Deliver your scanned solution as a UNIQUE pdf file (multiple files will not be accepted) and name it Surname-Name-exam.pdf. Send it to the following two email addresses: f.h.j.redig@tudelft.nl and fhjredig@gmail.com.

d) The exam is open book, i.e., you are allowed to use book and course notes. Notice that you are not allowed to do internet searches during the exam.

Please start your exam by writing down the code of honour sentence:
"I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."
An exam without this sentence will not be corrected.

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1. Are the following statements true or false? Justify (concisely) your answer (with an unjustified true or false answer, you do not earn points).

   a) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G}' \subset \mathscr{G} \subset \mathscr{F}$ are two sub-σ-algebras of $\mathscr{F}$. Then we have, for $X \in L^2(\Omega, \mathscr{G}', \mathbb{P}), Y \in L^2(\Omega, \mathscr{G}, \mathbb{P})$
   $$\mathbb{E}(XY|\mathscr{G}) = XY$$

b) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\mathscr{G} \subset \mathscr{F}$ is a sub-σ-algebra of $\mathscr{F}$. If X and Y are independent bounded random variables, and B is an event which is independent of $\mathscr{G}$, then we have

$$\mathbb{E}(XY1_B|\mathscr{G}) = \mathbb{P}(B)\mathbb{E}(X|\mathscr{G})\mathbb{E}(Y|\mathscr{G})$$

c) $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, and $\{\mathscr{F}_n, n \in \mathbb{N}\}$ is a filtration. Additionally, $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is another filtration, and we have for all $n \in \mathbb{N}$ $\mathscr{F}_n \supset \mathscr{G}_n$. Then every martingale w.r.t. the filtration $\{\mathscr{G}_n, n \in \mathbb{N}\}$ is also a martingale w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

d) The square root of a non-negative martingale is a supermartingale.

e) Let $\{Y_i, i \in \mathbb{N}\}$ be a sequence of i.i.d. standard normal random variables, and $\mathscr{F}_n = \sigma(Y_1, \dots, Y_n)$ is the associated natural filtration. Then

$$\tau = \inf\left\{k : \sum_{i=1}^{k} Y_i > 10\right\}$$

is a bounded stopping time w.r.t. the filtration $\{\mathscr{F}_n, n \in \mathbb{N}\}$.

f) An L^2-bounded martingale is a Doob martingale.

g) An L^3-bounded martingale converges almost surely and in L^1.

h) Let $\{W(t), t \geq 0\}$ denote Brownian motion. Then the process $\{W(1-t) - (1-t)W(1) : 0 \leq t \leq 1\}$ is a Brownian bridge.

i) Let $\{W(t), t \geq 0\}$ denote Brownian motion. Define

$$Z(t) = \sqrt{t}(W(1))$$

then $\{Z(t) : t \geq 0\}$ is also a Brownian motion.

j) Let (X_1, X_2) be a Gaussian vector. Assume that

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = 0.$$

Then the vector (Y_1, Y_2) given by

$$\begin{aligned} Y_1 &= \frac{X_1 + X_2}{\sqrt{2}} \\ Y_2 &= \frac{X_1 - X_2}{\sqrt{2}} \end{aligned}$$

is a Gaussian vector and has the same distribution as (X_1, X_2).

2. Let $\{Y_i, i \in \mathbb{N}\}$ denote an i.i.d. sequence of standard normal random variables.

   Additionally, we have a random variable X which is independent of the sequence $\{Y_i, i \in \mathbb{N}\}$ and which has the Bernoulli distribution, i.e., X takes values in $\{0, 1\}$ with $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0)$, where $0 < p < 1$.

   You are allowed to use that $\mathbb{E}e^{tY_i} = e^{\frac{t^2}{2}}$ for all $t \in \mathbb{R}$, and $\mathbb{E}(Y_i^4) = 3$. We denote by $\mathscr{F}_n = \sigma\{Y_i, 1 \leq i \leq n\}$ the natural filtration associated to the sequence $\{Y_i, i \in \mathbb{N}\}$. Moreover we denote $S_n = \sum_{i=1}^n Y_i$ for $n \geq 1$, and $S_0 = 0$

   a) Compute $\mathbb{E}(S_2^X | \mathscr{F}_1)$.

   b) Compute $\mathbb{E}(S_n^2 | \mathscr{F}_m)$ for $1 \leq m \leq n$.

   c) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Determine the Doob-Meyer decomposition of the submartingale $Z_n := (\sum_{i=1}^n a_i Y_i)^2$. (You do not have to show that Z_n is a submartingale.)

   d) Let $\{a_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Show that if $\sum_{i=1}^\infty a_i^2 < \infty$, then
   $$X_n = \sum_{i=1}^n a_i Y_i$$
   is a martingale which converges almost surely and in L^2.

   e) Denote for $a, b \in \mathbb{N}, a, b \geq 1$: $\tau_{a,b} = \inf\{n \in \mathbb{N} : S_n \in \{-a, b\}\}$. Show that $\{S_{n \wedge \tau_{a,b}}, n \in \mathbb{N}\}$ is a Doob martingale. If needed, you are allowed to use that $\mathbb{E}(\tau_{a,b}) < \infty$.

3. Let $\{W(t) : t \geq 0\}$ denote Brownian motion, $\mathscr{F}_t, t \geq 0$ its natural filtration. Denote, for $a, b \in (0, \infty)$:
   $$\tau_{a,b} = \inf\{t \geq 0 : W(t) \in \{-a, b\}\}, \tag{1}$$
   and
   $$T_a = \inf\{t \geq 0 : W(t) = a\} \tag{2}$$
   and finally
   $$\mathscr{T}_{a,b} = \inf\{t > \tau_{a,b} : W(t) = 0\} \tag{3}$$
   You are allowed to use that $T_a, \mathscr{T}_{a,b}, \tau_{a,b}$ are finite stopping times.

   a) Use an appropriate martingale to compute $\mathbb{E}e^{-\lambda T_a}$ for $\lambda > 0$.

b) Denote $M(t) = \sup_{0 \le s \le t} W(s)$. Compute $\mathbb{E}[(M(t) - W(t))^4]$. You are allowed to use that for a standard normal random variable Z, $\mathbb{E}(Z^4) = 3$.

c) Show that $\mathbb{E}\left([W(t \wedge \mathscr{T}_{a,b}) - W(t \wedge \tau_{a,b})]^2\right) = \mathbb{E}(t \wedge \mathscr{T}_{a,b} - t \wedge \tau_{a,b})$.

d) Show that with probability one for every $a > 0$, Brownian motion contains a zero in the interval $(0, a)$, i.e, there exists $0 < t < a$ such that $W(t) = 0$.

e) Let $\{W_b(t), 0 \le t \le 1\}$ denote Brownian bridge, i.e., for $0 \le t \le 1$, $W_b(t) = W(t) - tW(1)$. Compute the covariance between $W_b(t)$ and $W(t)$, for $0 \le t \le 1$.