Faculty of Electrical Engineering, Mathematics and Computer Science
Numerical Methods I, AM2060, BSc Applied Mathematics
Exam, June 26, 2020, 13:30 - 16:30

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Exam reviewer: M.B. van Gijzen.

Grade of exam = Pass if the sum of the points is larger than or equal to 11.6
and Fail if it is below 11.6
NB. This exam contains three questions!

1 We will analyse *Lagrange interpolation*. For given points $x_0, x_1, \ldots, x_n$ and their respective function values $f(x_0), f(x_1), \ldots, f(x_n)$, the interpolating polynomial $L_n(x)$ is given by

$$L_n(x) = \sum_{k=0}^{n} f(x_k) L_{kn}(x), \quad \text{with}$$
$$L_{kn}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

a *Determine* $\hat{L}_2(2)$ (the perturbed version of $L_2(2)$) given the following *measured* values:

k	x_k	$\hat{f}(x_k)$
0	1	3
1	3	6
2	4	5

(2 pt.)

b Given is that we know

$$|f(x) - \hat{f}(x)| \le \varepsilon,$$
$$|f'''(x)| \le \delta,$$

and

$$f(x) - L_n(x) = \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\zeta(x)),$$

for $x \in [1,4]$. Determine an *upper bound* for the error $|f(2) - \hat{L}_2(2)|$. (3 pt.)

2 We consider the generic initial value problem

$$y' = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1}$$

of which we approximate the solution by the following (implicit) predictor-corrector method

$$\begin{cases} w_* = w_n + \dfrac{\Delta t}{2} f(t_{n+1}, w_*), \\ \\ w_{n+1} = w_n + \Delta t f(t_n + \dfrac{1}{2}\Delta t, w_*). \end{cases} \tag{2}$$

a Use the test equation to prove that the local truncation error is of order $\mathcal{O}(\Delta t^2)$ (You may use the Geometric Series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$). (3pt)

b Derive a stability criterion for Δt and sketch the stability region in the complex plane. (2pt) c We consider the following system of differential equations

$$y_1' = -2y_1 + y_2,$$
$$y_2' = y_1 - 2y_2, \qquad (3)$$

which values of Δt give a stable numerical numerical integration to the above system if we use method (2)? (2pt)

d Does the numerical solution obtained by method (2) applied to problem (3) converge? Motivate your answer. (1pt)

e Compare the behaviour of numerical stability of method (2) to the stability of the Trapezoidal method, which reads as

$$w_{n+1} = w_n + \frac{\Delta t}{2}\left(f(t_n, w_n) + f(t_n + \Delta t, w_{n+1})\right). \qquad (4)$$

(2pt)

3 We consider the one-dimensional **convection-diffusion equation** with Dirichlet boundary conditions:

$$\begin{cases} -\varepsilon u'' + u' = 1, & 0 < x < 1, \\ u(0) = 0, & u(1) = 0, \end{cases} \qquad (5)$$

where $u = u(x)$, $u' = \frac{du}{dx}$ and $u'' = \frac{d^2u}{dx^2}$

a Show that

$$u(x) = x - \frac{1 - e^{x/\varepsilon}}{1 - e^{1/\varepsilon}} \qquad (6)$$

is the *exact solution* to the boundary value problem (5). (1 pt.)

b We solve the boundary value problem (5) using *central finite differences* for the diffusive term and *upwind finite differences* for the convective term.

For all *interior nodes* x_j the discretization method reads

$$-\varepsilon \frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + \frac{w_j - w_{j-1}}{\Delta x} = 1, \text{ for } j \in \{1, \dots, n\}, \qquad (7)$$

with $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size.

Give a *discretization method* for the two boundary nodes x_1 and x_n. (1 pt.)

c Give an expression of the local truncation error. (1 pt.)

d Use a step size of $\Delta x = 1/4$ to derive the system of equations $\mathbf{Aw} = \mathbf{f}$. Take care of the boundary conditions. The system must have three unknowns and three equations, i.e. $\mathbf{A}$ is a 3×3 matrix and $\mathbf{w}$ and $\mathbf{f}$ are three dimensional column vectors.

You do **not** have to solve this system. (1 pt.)

e Will the discretization method (7) produce oscillatory solutions? Motivate your answer. (1 pt.)