Exam Optimization (AM2020)

19-01-2021

13:30 – 16:00

This exam consists of 6 questions on 3 pages. You can earn 60 points in total. Your grade is determined by dividing the total number of obtained points by 6. Success!

- You only earn points when you use the right method and describe all steps and arguments clearly.
- You may use a calculator, the lecture notes and your own notes. You are **not allowed to collaborate** or to use any other sources.
- Include the following sentence: "This exam is solely undertaken by myself, without any assistance from others." followed by your name.
- Scan your solutions and your student ID and combine the scans into a **single pdf file**. Use a scanner or a scanner app.
- Upload the combined pdf in the assignment folder "Exam" on Brightspace before 16:15 (or 16:40 if you have extra time).
- During the exam, questions can be asked via email to l.j.j.vaniersel@tudelft.nl.
- After the exam, you may be invited for a **face-to-face check** on Wednesday 20-01-2021 between 14:45 and 16:45. Keep this time free and check your student email regularly.

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1. (14 points) Indicate for each of the statements below whether they are true (T) or false (F). For each correct answer you get 2 points, for each incorrect answer you get -1 points and for not filling in an answer you get 0 points.

   (i) If an LP has a feasible solution, then it has an optimal solution.

   (ii) A Steepest Descent sequence always converges to a global minimizer.

   (iii) If matrices A and B are both totally unimodular then the matrix $\begin{bmatrix} A & B \end{bmatrix}$ is also totally unimodular.

   (iv) A 3-approximation algorithm for a minimization problem can return a solution with value 16 for an instance with optimal value 8.

   (v) The matrix below is positive definite.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(vi) If $\boldsymbol{x}^*$ is a critical point of a convex function $f(\boldsymbol{x})$ with continuous first and second partial derivatives, then $\boldsymbol{x}^*$ is a global minimizer for $f(\boldsymbol{x})$.

(vii) The function $f(x_1, x_2) = (x_1 + 2x_2)^4$ is coercive.

2. (5 points) **Prove**, without using the Simplex algorithm, that the following LP has a bounded optimum.

$$\begin{array}{lrcrcrcl} \min & z = & x_1 & - & x_2 & + & 2x_3 & \\ \text{s.t.} & & 3x_1 & - & 3x_2 & + & 4x_3 & \leq 5 \\ & & -x_1 & + & 2x_2 & - & 2x_3 & \leq 7 \\ & & \multicolumn{5}{l}{x_1,\ x_2,\ x_3 \geq 0} & \end{array}$$

3. (10 points) Consider the problem of scheduling E exams when only a limited number L of students is allowed to be on campus simultaneously. For each exam, we need to decide whether it takes place on-line or on-campus. If it takes place on-campus, you also need to select a time-slot. There are T (disjoint) time-slots. There are s_e students who signed up for exam e. Assume that, if an exam is on-campus, all students who signed up make the exam on-campus. Similarly, if an exam is on-line, all students who signed up make the exam on-line. Finally, assume that each student signed up for exactly one exam. The goal is that a maximum number of students make their exam on-campus. Give an **ILP formulation** of this problem.

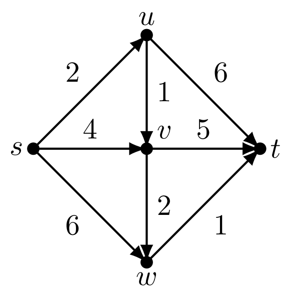
4. (10 points) **Prove** that the following problem is NP-complete.

   HITTING
   Given: a set U, a collection $S = \{S_1, \dots, S_r\}$ of subsets of U (so $S_i \subseteq U$ for $i = 1, \dots, r$) and an integer k
   Decide: can we select at most k elements of U such that each of $S_1, \dots, S_r$ contains at least one selected element?

   You may use that the following problem is NP-hard. Here, $N^+(v)$ is the set containing vertex v and all its neighbours.

   DOMINATION
   Given: a graph $G = (V, E)$ and an integer k'
   Decide: can we select at most k' vertices from V such that, for each $v \in V$, at least one element of $N^+(v)$ is selected?

5. Consider the following directed graph in which the label of each arc a indicates its length.



   (a) (5 points) Find the **length** of a shortest s-t path using the algorithm of **Dijkstra**.

   (b) (5 points) Use **Complementary Slackness** to **find** a shortest s-t path and to **prove** that no shorter s-t path exists.

(c) (5 points) Now interpret the labels of the arcs as capacities and **prove**, using a theorem discussed in this course, that no s-t flow of value 8 exists.

6. (6 points) Consider the following Simplex tableau of the LP relaxation of a maximization ILP.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
s_1	17/2	-5/2	0	2	1	-1/3
x_2	7/2	1/2	1	-2	0	-1/2
$-z$	7/2	-1/2	0	0	0	-1/2

Find the **Gomory cutting plane** corresponding to the first row of the tableau. You do not need to express it in the original variables x_1, x_2, x_3.