Solutions Exam 27 january 2020

1. $t^2x'' + tx' + x = 1+t \quad t>0$

homogeneous solution : $x_h(t)$

a) This an Euler equation so we can try $x_h(t) = ct^r$

$$c\,[r(r-1) + r + 1\,]\,t^r = 0$$

$$r^2 - r + r + 1 = 0 \quad r^2 + 1 = 0 \iff r = \pm i$$

solutions are $t^i = e^{i\ln t} = \cos(\ln t) + i\sin(\ln t)$

real and imaginary parts are also solutions,

hence $\quad x_h(t) = c_1\cos(\ln t) + c_2\sin(\ln t)$

b) try polynomial $x(t) = at + b$ since right-hand side is linear in t.

$$at + at + b = 1 + t \quad \rightarrow \quad b = 1$$

and $a = \frac{1}{2}$

c)

$$x(t) = c_1\cos(\ln t) + c_2\sin(\ln t) + \tfrac{1}{2}t + 1$$

$$x(1) = c_1 + \tfrac{3}{2} = 1 \qquad c_1 = -\tfrac{1}{2}$$

$$x'(1) = \tfrac{1}{2} + c_2\cos(\ln t)\,\tfrac{1}{t}\Big|_{t=1} = \tfrac{1}{2} + c_2 = 1$$

$$x(t) = -\tfrac{1}{2}\cos(\ln t) + \tfrac{1}{2}\sin(\ln t) + \tfrac{1}{2}t + 1$$

2. $\dot{\underline{x}} = A\underline{x}$

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} = -I + \underbrace{\overbrace{\begin{pmatrix} 0 & 2 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}^{N}}_{\text{nillpotent.}}$$

a fundamental matrix is

$$e^{At} = e^{(-I+N)t} = e^{-It} e^{Nt}$$

$$= e^{-t}\left(I + Nt + \tfrac{1}{2} N^2 t^2 \right) \qquad [N^3 = 0]$$

$$= e^{t} \begin{pmatrix} 1 & 2t & -3t+2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}$$

· Solution of $\dot{\underline{x}} = A\underline{x}$ with $x(0) = x_0$

is $\underline{x}(t) = e^{At}\underline{x}_0$ and this

solution exists for all $t \in \mathbb{R}$!

3. equilibria : There is only one namely $(0,0)$

a) If $x=0$, it automatically follows that $\dot{x}=0 \Rightarrow y=0$
and if $y=0$ .. .. . $\dot{y}=0 \Rightarrow x=0$

if $x \neq 0$ and $y \neq 0$ then $\dot{y}=0 \Rightarrow x=-y f(x,y)$

and therefor $\dot{x}=0=-y[1+f^2(x,y)]$.

$\Rightarrow y=0$, which is not possible.

b) linearize about the origin 0.

f satisfies all requirements of the theorem of linearizing the differential equation. $f(0,0)=0$ and

$$\lim_{x \to 0} \frac{\|f(x,y)\|}{\|x\|} = 0$$

$$Df|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\dot{u} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u$ is linearized system.

eigenvalues $\lambda = \pm i$, dus $(0,0)$ is centrum{center} And is stable

As there is no real part, stability of O does not tell any thing about the stability of the O in the nonlinear system (1)

3c) $\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = -r\sin\theta + r\cos\theta\, f(r,\theta)$

$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = r\cos\theta + r\sin\theta\, f(r,\theta)$

multiply $\dot{x}$ by $\cos\theta$ and $\dot{y}$ by $\sin\theta$ and add:

$$\dot{r} = r\, f(r,\theta) = r^3 \sin\left(\frac{\pi}{r}\right) \qquad (r>0)$$

$r=0$ is an equilibrium point, so you do not have to calculate this explicitly.

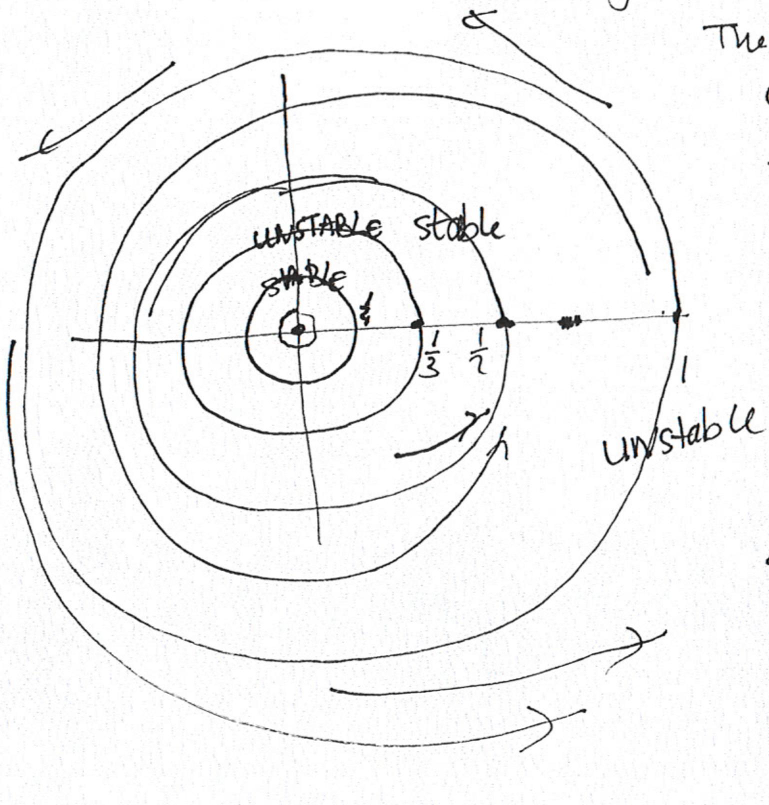
Next multiply $\dot{x}$ by $-\sin\theta$ and $\dot{y}$ by $\cos\theta$ and add gives

$r\dot{\theta} = r$, which reduces to $\dot{\theta} = 1$ for $r>0$

3d) Limit cycles can be calculated by setting $\dot{r} = 0$

hence $\sin\left(\frac{\pi}{r}\right) = 0$, which yields $r = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

so there infinitely many limit cycles.



The stability of the limit cycles can be determined from the sign of $\dot{r}$ near $r=1$, if we decrease r then $\dot{r} < 0$ and if $r>1$ $\dot{r} > 0$, so $r=1$ is unstable. The other limit cycles alternatingly are stable / unstable

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$$y'' + y = \begin{cases} t^2 & 0<t<1 \\ 0 & t \geq 1 \end{cases} \qquad y(0) = y'(0) = 0$$

$$\mathcal{L}[y''+y] \overset{①}{=} s^2\hat{y} + \hat{y} = \mathcal{L}[(1 - H_1(t))(t^2)]$$

$$= \mathcal{L}[t^2] - \mathcal{L}[H_1(t)t^2] = \mathcal{L}[t^2] - \mathcal{L}[H_1(t)[(t-1)^2 + 2(t-1) + 1]]$$

① $$= \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$\hat{y}(s) = \frac{2}{s^3}\frac{(1-e^{-s})}{(1+s^2)} - \frac{2e^{-s}}{s^2(1+s^2)} - \frac{e^{-s}}{s(1+s^2)}$$

$$= \cancel{2}(1-e^{-s})\left[\frac{2s}{s^2+1} - \frac{2(s^2-1)}{s^3}\right] - e^{-s}\left[\frac{2}{s^2} - \frac{2}{1+s^2}\right]$$

①

$$- e^{-s}\left[-\frac{s}{1+s^2} + \frac{1}{s}\right]$$

you can now use the table to find the inverse Laplace transforms.

$$y(t) = \mathcal{L}^{-1}[\hat{y}(s)] = 2\cos t - 2\cos(t-1)H_1(t)$$

$$+ t^2 - (t-1)^2 H_1(t) - 2(1 - H_1(t))$$

$$- 2H_1(t)(t-1) + 2H_1(t)\sin(t-1)$$

$$+ H_1(t)\cos(t-1) - H_1(t)$$

$$= 2\cos t + t^2 - 2 + H_1(t)[2\sin(t-1) - \cos(t-1) + 1$$

$$- 2(t-1) - (t-1)^2]$$ ②

5. $\dot{y} = A(t)\, y$ (3) $\quad$ $\underline{x}(t)$ is a solution.

$y(t) = \phi(t)\, \underline{x}(t) + \underline{z}(t)$

with $\underline{z}(t) = [0, z_2(t), \ldots z_n(t)]^T$

a) substitute $y(t) = \phi(t)\, \underline{x}(t) + \underline{z}(t)$ in (3)

$$\dot{y} = \dot{\phi}\, \underline{x}(t) + \underbrace{\phi(t)\, \dot{\underline{x}}} + \dot{\underline{z}} = A[\underbrace{\phi(t)\, \underline{x}}] + A\, \underline{z}(t)$$

are the same as $\underline{x}(t)$ is a solution of $\dot{\underline{x}} = A\underline{x}$

therefore $\dot{\underline{z}} = A\underline{z} - \dot{\phi}\, \underline{x}$. $\quad$ (*)

b) First component of $z_1(t) = 0$, therefore we have

from (*) : $\dot{z}_1(t) = 0 = (A\underline{z})_1 - \dot{\phi}\, x_1(t)$

$$= \sum_{j=2}^{n} A_{1j}\, z_j - \dot{\phi}\, x_1(t) = 0$$

Hence it is ok for $z_1(t)$

All the other components of $\underline{z}(t)$ are

immediate : $\dot{z}_i(t) = \sum_{j=2} A_{ij} z_j - \dot{\phi}\, x_i(t)$

c) The equation for $\phi(t)$ follows immediately from (b) first component.

$$\dot{\phi} = \frac{1}{x_1(t)} \sum_{j=2}^{n} A_{1j} z_j$$

or $\phi(t) = \int \frac{1}{x_1(t)} \sum_{j=2}^{n} A_{1j}\, z_j(t)$