

Mid term Exam Ordinary Differential Equations, AM2030

Tuesday 10 December 2019, 13.30-16.30h

- This exam consists of 5 regular problems and one bonus problem.
 - All answers need to be justified.
 - Norm: total of 35 points; the distribution of points is as shown in the exercises. The exam grade is (total points)/3.5.
 - *Success !*
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1. The following differential equation for the function $x : \mathbb{R} \rightarrow \mathbb{R}$ is given

$$\frac{dx}{dt} = x^2 + 1.$$

- a. (3) Find the solution $x(t)$ using initial condition $x(0) = 1$, and give the maximal interval of existence of the solution.

The differential equation for the function $y : \mathbb{R} \rightarrow [1, \infty)$ reads

$$\frac{dy}{dt} = t^2 \sqrt{y-1}.$$

- b. (3) Find a solution $y(t)$ using initial condition $y(0) = 1$, and argue whether this solution is unique or not.

2. The differential equation for the function $y = y(x)$ is

$$(x-1)^2 y'' + 2x(x-1)y' + (x-1)y = 0, \quad x > 1. \quad (1)$$

- a. (1) Show that $x = 1$ is a regular singular point of equation (1).
b. (4) Give the indicial equation $F(r) = 0$ and find the two solutions for r and write the recurrence relation for the coefficients a_n (**You do not need to solve it!**)
c. (2) Determine the first 3 nonzero terms of the series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^{n+r},$$

only for the **largest root** of the indicial equation found in (b).

- d. (2) What can be said about the radius of convergence of this series?
e. (2) How would you determine a second solution of (1) linearly independent solution of the one found in c)?

You do not need to do calculation, giving the method is sufficient!

3. Consider the following differential equation for $y(x)$:

$$(3y^2 + 8x^2y) + (3xy + 2x^3)\frac{dy}{dx} = 0$$

- a. (2) Is the differential equation exact?
- b. (4) Determine the general solution of this differential equation. If necessary find an integrating factor.

4. The following differential equation for $y(t)$ is given:

$$y'' + 9y = \sin(3t) + e^{9t} \quad (2)$$

- a. (2) Find the solution of the homogeneous problem.
- b. (2) Find the general solution of (2).
- c. (2) Assume that the initial conditions are $y(0) = 1, y'(0) = 0$, give the solution of the initial value problem.

5. (6) A differential equation for the function $y(t)$ with initial conditions is given as

$$y'' + y = t^2 + 1, \quad y(0) = 0, y'(0) = 0. \quad (3)$$

Find the solution of this differential equation **using the Laplace transform!**

BONUS: *Generalization of Gronwall's inequality*

6. (3.5) Show that if $g : [0, T] \rightarrow \mathbb{R}$ is continuous and if there exist nonnegative constants C, B, K , such that

$$g(t) \leq C + Bt + K \int_0^t g(s) ds, \quad 0 \leq t \leq T$$

then

$$g(t) \leq Ce^{Kt} + B \frac{e^{Kt} - 1}{K} \quad 0 \leq t \leq T$$

Hint: You can use the same techniques as we used to prove uniqueness of solutions.