1. a.

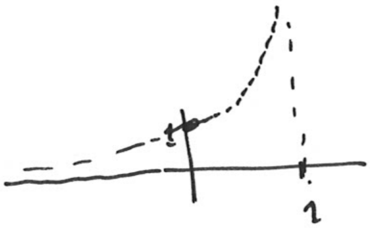
$$\dot{x} = x^2 \qquad x(0) = 1$$

separable differential equation.

$$\int_1^x \frac{d\tilde{x}}{\tilde{x}^2} = \int_0^t 1 \, d\tilde{t} = t$$

$$-\frac{1}{\tilde{x}} \Big|_1^x = t$$

$$-\frac{1}{x} + 1 = t \quad , \text{ hence } \quad x(t) = \frac{1}{1-t}$$



solution exists for all $t \in (-\infty, 1)$

1b)

$$\frac{dy}{dt} = \sqrt{y-1} \qquad y(0) = 1$$

$y(t) = 1$, is a solution for all $t \in \mathbb{R}$.

This solution is not unique, however., since

$$\int_1^y \frac{d\tilde{y}}{\sqrt{\tilde{y}-1}} = \int_0^t dt = 2\sqrt{\tilde{y}-1} \Big|_1^y = t$$

$$2\sqrt{y-1} = t$$

$$y(t) = 1 + \left(\frac{t}{2}\right)^2$$ is also a solution.

for all $t \geq 0$

2. $x^2 y'' + 2xy' - \alpha^2 x^2 y = 0$ $y(0)=1$ $y'(0)=0$ $\alpha \in \mathbb{R}$

a. $x=0$ is a singular point, as the coefficient for y'' vanishes for $x=0$.

It is a regular singular point since : $y'' + \frac{2y'}{x} - \alpha^2 y = 0$ (~~...~~

and the theorem says that if $y'' + p(x)y' + q(x)y = 0$
where $p(x) = \frac{2}{x}$; $q(x) = -\alpha^2$

and $xp(x)$ ~~is~~ analytic, and $x^2 q(x)$ analytic } then point $x=0$ is regular singular.

$x \cdot \frac{2}{x} = 2$ is analytic and $-\alpha^2 \cdot x^2$ is analytic too., hence $x=0$ is regular singular.

b. Indicial equation.

$r(r-1) + p_0 r + q_0 = 0$ $p_0 = 2$. $q_0 = 0$

Hence $r(r-1) + 2r = 0$

$r^2 + r = 0 \rightarrow r=0$ or $r=-1$

c, substitute $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ in Eq(1) $(a_0 \neq 0)$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \alpha^2 \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

shift the index in the last term by 2 :

$$-\alpha^2 \sum_{n=0}^{\infty} a_n x^{n+r+2} = -\alpha^2 \sum_{n=2}^{\infty} a_{n-2} x^{n+r},$$

which yields equal powers of x in all summations :

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \alpha^2 \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

The last sum starts at $n=2$, so we have for $n \geq 2$

$$a_n (n+r)(n+r-1) + 2a_n (n+r) - \alpha^2 a_{n-2} = 0.$$

the $n=0$, and $n=1$ we will take separately

$n=0$: $\quad a_0 \, r(r-1) + 2a_0 r = 0 \rightarrow$ this is the indicial equation again!
$a_0 \neq 0$

$n=1$ $\quad a_1 (r+1) r + 2 a_1 (r+1) = 0$

$$(r+1) a_1 (r+2) = 0.$$

since we know that $r=0$ or $r=-1$ from the indicial equation.

this gives $a_1 = 0$ if $r=0$
and a_1 undetermined if $r=-1$

d. We next solve the recurrence relation for

$r=0$: $\quad a_1 = 0$ and $a_n = \dfrac{\alpha^2 a_{n-2}}{n(n+1)} \quad n \geq 2$

$$a_2 = \frac{\alpha^2 a_0}{2.3}$$

$$a_4 = \frac{\alpha^2 a_2}{4.5} = \frac{\alpha^4 a_0}{2.3.4.5}$$

$\vdots$

$$a_{2n} = \frac{\alpha^{2n} a_0}{2.3.4.5.(2n+1)}$$

the odd terms are all 0, since $a_1 = 0$.

This gives a solution $y_1(x) = a_0 \sum_{n=0}^{\infty} \dfrac{\alpha^{2n} x^{2n}}{(2n+1)!}$

next we solve for ~~ϕ~~ $r=-1$

$$a_n = \frac{\alpha^2 a_{n-2}}{(n-1)n}. \qquad n \geq 2. \quad , a_1 \text{ undetermined.}$$

$$a_2 = \frac{\alpha^2 a_0}{2 \cdot 1}$$

$$a_4 = \frac{\alpha^2 a_2}{4 \cdot 3} = \frac{\alpha^4 a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$a_{2n} = \frac{\alpha^{2n}}{(2n)!} a_0$$ This gives a solution.

$$y_2(x) = a_0 \left(\sum_{n=0}^{\infty} \frac{\alpha^{2n} x^{2n}}{(2n)!} \right) x^{-1}$$

the odd terms:

$$a_3 = \frac{\alpha^2 a_1}{3 \cdot 2}$$

$$a_5 = \frac{\alpha^2 a_3}{5 \cdot 4} = \frac{\alpha^4 a_1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$a_{2n+1} = a_1 \frac{\alpha^{2n}}{(2n+1)!},$$ this gives a solution

$$y_3(x) = a_1 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n+1)!} x^{2n+1} \cdot x^{-1} = a_1 \sum_{n=0}^{\infty} \frac{\alpha^{2n} x^{2n}}{(2n+1)!}$$

which is a multiple of $y_1(x)$, so we need not consider this. As $y'(0) = 0$, we also do not include $y_2(x)$, as this solution blows up if $x \downarrow 0$. This implies that the solution is given as: $y(x) = \sum_{n=0}^{\infty} \frac{\alpha^{2n} x^{2n}}{(2n+1)!}$, that indeed satisfies $y(0) = 0.1$ and $y'(0) = 0$.

3b

$$\dot{x} = A x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t \sin t$$

Make the equation complex : $\dot{\underline{z}} = A\underline{z} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{(1+i)t}$

and then take imaginary part of solution $\underline{z}(t)$

We try $\underline{z}(t) = \underline{b}\, e^{(1+i)t}$

$$\dot{\underline{z}} = \underline{b}(1+i) \cancel{e^{(1+i)t}} = A \underline{b} \cancel{e^{(1+i)t}} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cancel{e^{(1+i)t}}$$

$$[A - (1+i)I]\underline{b} = -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ write augmented matrix

$$\left(\begin{array}{ccc|c} -2-i & a & 0 & 0 \\ 0 & -2-i & a & -1 \\ 0 & 0 & -2-i & 0 \end{array}\right) \to$$

$b_3 = 0$. $\quad -(2+i) b_2 + a \underline{b_3} = -1$

$$b_2 = \frac{1}{(2+i)}$$

$$b_1 = \frac{a b_2}{(2+i)} = \frac{a}{(2+i)^2}$$

Hence $\underline{b} = \begin{pmatrix} \frac{a}{(2+i)^2} \\ \frac{a}{(2+i)} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{a}{3+4i} \\ \frac{a}{2+i} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{a(3-4i)}{25} \\ \frac{a(2-i)}{5} \\ 0 \end{pmatrix}$

Hence $\underline{x}(t) = \mathrm{Im}\left[\underline{b}\, e^{(1+i)t}\right] = \mathrm{Im}\left\{\frac{a}{25} e^t \begin{pmatrix} 3-4i \\ 10-5i \\ 0 \end{pmatrix} [\cos t + i \sin t]\right\}$

$$= \frac{a}{25} e^t \begin{bmatrix} 3\sin t - 4\cos t \\ 10 \sin t - 5\cos t \\ 0 \end{bmatrix}.$$

4 : $\dot{x} = (x^2-1)\,y$
$\dot{y} = (x+2)(y-1)(y+2)$

a) equilibra: $\dot{x} = 0 \rightarrow x = \pm 1$ or $y = 0$ → gives $x = -2$ when substituting $y=0$

substitute in $\dot{y} = 0$

gives $y = 1$ or $y = -2$

So we have 5 points : $(1,1)$ $(-1,1)$

$(1,-2)$ $(-1,-2)$

$(-2,0)$

b) calculate Jacobian matrix. DF

$$DF = \begin{pmatrix} 2xy & (x^2-1) \\ (y-1)(y+2) & (x+2)[2y+1] \end{pmatrix}$$

substitute : $(1,1)$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 9 \end{pmatrix}$$ $\lambda_1 = 2$, $\lambda_2 = 9$ unstable node.

$(-1,1)$: $$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$ $\lambda_1 = -2$, $\lambda_2 = 3$ → saddle. (unstable)

$(1,-2)$: $$\begin{pmatrix} -4 & 0 \\ 0 & -9 \end{pmatrix}$$ $\lambda_1 = -4$, $\lambda_2 = -9$ asympt. stable node

$(-1,-2)$ $$\begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix}$$ $\lambda_1 = 4$, $\lambda_2 = -3$ saddle point

$(-2,0)$ $$\begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}$$ $\lambda^2 + 6 = 0$, $\lambda = \pm i\sqrt{6}$ center. stable

4c) All equilibria of the linear system and their stability carry over to the nonlinear system. ~~However~~, except the center point, which has $Re(\lambda) = 0$, and therefore stability cannot be decided using linearization.

4d) orbits: $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(x+2)(y-1)(y+2)}{(x+1)(x-1)\,y}$

$$\int \frac{y\,dy}{(y-1)(y+2)} = \int \frac{(x+2)}{(x+1)(x-1)}\,dx$$

$$\int \left\{ \frac{1}{y+2} + \frac{1}{3}\left[-\frac{1}{y+2} + \frac{1}{y-1} \right] \right\} dy = \int \left\{ \frac{1}{x-1} + \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1} \right) \right\} dx$$

$$\int \left\{ \frac{2}{3}\frac{1}{y+2} + \frac{1}{3}\frac{1}{y-1} \right\} dy = \int \left(\frac{3}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1} \right) dx$$

$$\frac{2}{3}\ln|y+2| + \frac{1}{3}\ln|y-1| = \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + k.$$

$$\ln\left[(y+2)^2|y-1|\right] = \frac{9}{2}\ln|x-1| - \frac{3}{2}\ln|x+1| + 3k$$

Hence: $\ln (y+2)^2|y-1| - \ln\left[\frac{|x-1|^{9/2}}{|x+1|^{3/2}}\right] = 3k.$

$$\ln\left[\frac{(y+2)^2|y-1||x+1|^{3/2}}{|x-1|^{9/2}}\right] = 3k$$

Hence we have:

$$F(x,y) = \frac{(y+2)^2|y-1||x+1|^{3/2}}{|x-1|^{9/2}} = C \geq 0$$

describes the orbits in implicit form.

4e) Since $F(x,y)$ depends on the absolute values we restrict ourselves to the region $: R \cdot \{(-\infty,-1) \times (-1,1)$

$x \leq -1$ and $y \in (-1,1)$

$\downarrow$ $\downarrow$

so $|x-1| = -x+1$ so $|y-1| = 1-y$

and $|x+1| = -x-1$

$F(x,y)$ on R is given as:

$$F_R(x,y) = (y+2)\frac{(1-y)\ (-x-1)^{3/2}}{(1-x)^{9/2}} = C.$$

To determine periodic orbits, we have to find intersections with the x-axis and the line $x=-2$.

First substitute $x=-2$:

$$F_R(-2,y) = \frac{(y+2)(1-y)\ 1}{3^{9/2}} = C.$$

$$+ y^2 + y - 2 + 81\sqrt{3}\,C = 0.$$

intersection points: $y_{1,2} = \frac{-1 \pm \sqrt{1-4(81\sqrt{3}\,C-2)}}{2}$

there will be 2 points if $-4(81\sqrt{3}\,C-2) \leq 1$

$81.4\sqrt{3}\,C \geq 7.$

Hence $C \geq \frac{7}{81.4\sqrt{3}}$

These points have to be 1 one positive, 1 negative, hence $C \leq \frac{2}{81\sqrt{3}}$

Next find 2 intersection points with $y=0$.

$$F_R(x,0) = \frac{2(-x-1)^{3/2}}{(1-x)^{9/2}}$$

You can see that $F_k(x,0) \to 0$ if $x \uparrow -1$

We now calculate $F_k'(x,0)$

$$F_k'(x,0) = \frac{3}{\cancel{2}}\cancel{2}\,\frac{(-x-1)^{1/2}\cdot(-1)}{(1-x)^{9/2}} + \cancel{2}(-x-1)^{3/2}(1-x)^{-11/2}\left(\frac{-9}{\cancel{2}}\right)\cdot(-1)$$

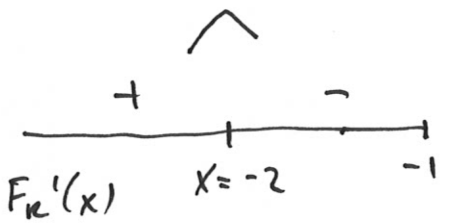
$$-3\,\frac{(-x-1)^{1/2}}{(1-x)^{11/2}}(1-x) + \frac{(-x-1)(-x-1)^{1/2}}{(1-x)^{11/2}}\,9$$

$$= \frac{(-x-1)^{1/2}}{(1-x)^{11/2}}\left[-3(1-x)+9(-x-1)\right]$$

so $F_k'(x,0)=0$ if $3x - 9x - 3 - 9 = 0$

$-6x = 12$

$x = -2$.



$F_k(x,0)$ has a maximum in $x=-2$. since $F_k''(-2,0) \neq 0$.

$$F_k(-2,0) = \frac{2\cdot 1}{81\sqrt{3}} = \frac{2}{81\sqrt{3}}$$

but $\frac{7}{4}\left(\frac{1}{81\sqrt{3}}\right) < C < 2\left(\frac{1}{81\sqrt{3}}\right)$

So indeed there are two points of intersection of $F_k(x,0)$ with C.

Since the solution curves are continuous there have to be infinitely many closed curves surrounding $(-2,0)$. There are no equilibria on this closed curve. Therefore the solution corresponding to the closed curve is periodic.

4f) From b,c) you know the saddles $(-1,1)$ and $(-1,-2)$ and an asymptotically stable node $(1,-2)$ and an unstable node $(1,1)$

the eigenvectors are always $(1,0)$ and $(0,1)$

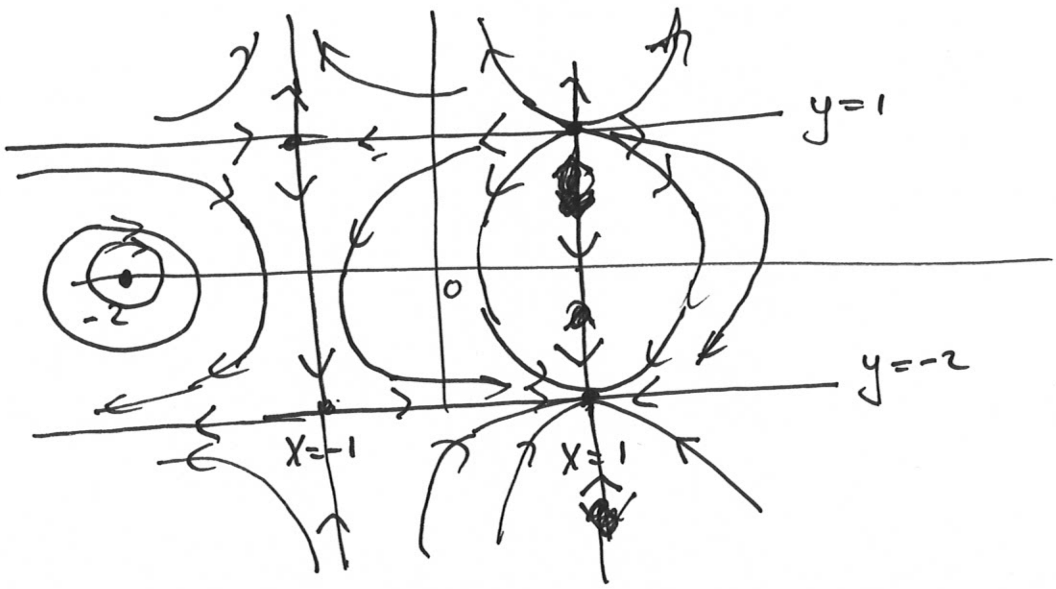
. Furthermore, we know from 4e) that there are periodic solutions around $(-2,0)$

The only new information that we can obtain by ~~calculatar~~ calculating some orbits

$x = \pm 1$ is an orbit

$\left.\begin{array}{l} y = 1 \\ y = -2 \end{array}\right\}$ are also orbits

This gives



5. $\dot{\underline{x}} = A(t)\,\underline{x}$ $\qquad \underline{x}(t) \in \mathbb{R}^n$

$\dot{\underline{X}} = A(t)\,\underline{X}$ $\qquad \underline{X}(t)$ is a fundamental matrix

$\dot{\underline{X}}^T = \underline{X}^T A^T$ by taking transpose

next write $(X^T)^{-1} X^T(t) = id.$

and differentiate wrt t.

$$\frac{d}{dt}\left[(X^T)^{-1} X^T\right] = \frac{d(X^{-1})^T}{dt}\,\underline{X}^T + (X^T)^{-1}\frac{d\underline{X}^T}{dt}$$

$$Y(t) := (X^{-1})^T(t)$$

$$\frac{dY}{dt}\,X^T(t) = -\underbrace{(X^T)^{-1}\,(X^T)}_{id.}\,A^T(t)$$

$$\frac{dY}{dt}\,(X^T)(t) = -A^T(t)$$

multiply by $(X^T)^{-1}(t)$

$$\frac{dY}{dt} = -A^T(t)\,(X^T(t))^{-1} = -A^T(t)\,\underline{Y}(t)$$