1 $\dot{x} = x^2+1$

a) $\displaystyle\int_1^x \frac{d\tilde{x}}{\tilde{x}^2+1} = \int_0^t d\tilde{t} = t$

$\arctan x - \arctan 1 = t$

$\arctan x = t + \frac{\pi}{4}$

$x = \tan\left(t+\frac{\pi}{4}\right)$ interval of existence $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$

b) Find a solution for $\dot{y} = x^2\sqrt{y-1}$

$$\int_1^y \frac{dy}{\sqrt{y-1}} = \int_0^t t^2\,dt$$

$$2\sqrt{y-1} = \frac{t^3}{3} + \cancel{c}$$

$$\sqrt{y-1} = \frac{t^3}{6} + \cancel{c}$$

$$y-1 = \left(\frac{t^3}{6} + \cancel{c}\right)^2$$

$$y(t) = 1 + \left(\frac{t^3}{6} + \cancel{c}\right)^2 = 1 + \frac{t^6}{36}$$

another solution is $y(t) = 1$

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2. $y'' + \frac{2x}{x-1}y' + \frac{y}{(x-1)} = 0$ (*)

a) $p(x) = \frac{2x}{x-1}$ and $q(x) = \frac{1}{x-1}$

Hence $x=1$ is a singular point of (*) because $\lim_{x\downarrow 1} p(x) = \infty$ and $\lim_{x\downarrow 1} q(x) = \infty$.

The point is regular singular as $\left(\lim_{x\to 1}\right)(x-1)p(x) = \left(\lim_{x\downarrow 1}\right)2x$ and $\left(\lim_{x\downarrow 1}\right)(x-1)^2 q(x) = \boxed{\lim_{x\downarrow 1}}(x-1)$

b) $y''(x)\,(x-1) + 2x\, y'(x) + y(x) = 0$

expand $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^{n+r}$ and write $w = x-1$

for convenience $\quad [\, 2x = 2(x-1) + 2 = 2w+2 \,]$.

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, w^{n+r-1} + 2\sum_{n=0}^{\infty} a_n w^{n+r} (n+r)$$

$$+ 2\sum_{n=0}^{\infty} a_n (n+r)\, w^{n+r-1} + \sum_{n=0}^{\infty} a_n w^{n+r} = 0$$

$$w^{r-1}\left[a_0\, r(r-1) + 2a_0 r\right] + \sum_{\boxed{n=1}}^{\infty} a_n (n+r)(n+r-1)\, w^{n+r-1}$$

$$+ 2\sum_{n=0}^{\infty} a_n w^{n+r} (n+r) + 2\sum_{\boxed{n=1}}^{\infty} a_n (n+r)\, w^{n+r-1} + \sum_{n=0}^{\infty} a_n w^{n+r} = 0$$

shift the terms that start with $\boxed{n=1}$ by 1; this gives

$$w^{r-1} a_0 \left[r^2 + r\right] + \sum_{n=0}^{\infty} a_{n+1} [n+1+r][n+r]\, w^{n+r}$$

$$+ 2\sum_{n=0}^{\infty} a_n w^{n+r} (n+r) + 2\sum_{n=0}^{\infty} a_{n+1} (n+1+r)\, w^{n+r} + \sum_{n=0}^{\infty} a_n w^{n+r} = 0$$

this again yields the $\boxed{\text{indicial equation}}$ for the terms with powers w^{r-1}: $\quad \left(r(r+1) = 0 \Rightarrow r = 0 \vee r = -1 \right)$

recurrence relation : $\quad a_{n+1} [n+1+r][n+r+2] = -a_n (2n+2r+1)$

$$a_{n+1} = -a_n \frac{(2n+2r+1)}{(n+1+r)(n+r+2)} \quad ; \; n = 0, 1, 2, \dots$$

For $r=0$ this gives : $\quad a_{n+1} = -a_n \dfrac{[2n+1]}{(n+1)(n+2)} \qquad n = 0, 1, \dots$

For $r=-1$ " " $\quad a_{n+1} = -a_n \dfrac{[2n-1]}{n(n+1)} \qquad n = 0, 1, 2.$

For $r=0$:

$$a_{n+1} = -\frac{a_n\,(2n+1)}{(n+1)(n+2)}$$

2c)

$$a_1 = -\frac{a_0}{2}$$

$$a_2 = -\frac{a_1\,3}{3.2} = \frac{3\,a_0}{3.2.2.}$$

$$a_3 = -\frac{a_2\,(5)}{3.4} = -\frac{5.3.\,a_0}{4.3.3.2.2.}$$

~~to~~ set $a_0 = 1$ for example

$$y(x) = 1\,(x-1)^0 + -\tfrac{1}{2}(x-1) + \tfrac{1}{4}(x-2)^2$$

$$= \tfrac{\cancel{3}}{2} - \tfrac{1}{2}x + \tfrac{1}{4}\left[x^2 - 4x + 4\right] = \tfrac{5}{2} - \tfrac{3}{2}x + \tfrac{1}{4}x^2$$

2d) The series converges for alle $x \in \mathbb{R}$, as $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 = R^{-1}$

For $r=-1$, we find: $a_1 \cdot 0 = +a_0$

$a_0 \neq 0 \rightarrow$ so this does not have a solution.

e) Theorem 2.8 states that a second solution has the form $y(x) = \sum_{n=0}^{\infty} b_n(r_2)(x-1)^{n+r_2} + \ln(x-1)\,y_1(x)$

Another way to find the solution is by order reduction

Both answers are fine.

3 $\underbrace{(3y^2+8x^2y)}_{M(x,y)} + \underbrace{(3xy+2x^3)}_{N(x,y)}\frac{dy}{dx} = 0$

a) $\frac{\partial M}{\partial y} = 6y + 8x^2$ $\qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so not exact!

$\frac{\partial N}{\partial x} = 3y + 6x^2$

b) Integrating factor: Try $\mu = \mu(x)$

$$\mu M_y = \frac{\partial[\mu M]}{\partial y} = \frac{\partial}{\partial x}[\mu N] = \mu_x N + \mu N_x$$

$$\mu_x = \mu\frac{[M_y - N_x]}{N} = \mu\left[\frac{6y+8x^2-3y-6x^2}{3xy+2x^3}\right] =$$

$$\mu\frac{[3y+2x^2]}{x[3y+2x^2]} = \frac{\mu}{x}.$$

Hence $\mu(x) = x$.

$$\frac{\partial\phi}{\partial x} = x[3y^2 + 8x^2y]$$

$$\frac{\partial\phi}{\partial y} = 3x^2y + 2x^4 \Rightarrow \phi(x,y) = \frac{3}{2}x^2y^2 + 2x^4y + h(x)$$

$$\frac{\partial\phi}{\partial x} = \cancel{3xy^2} + 8\cancel{x^3y} + h'(x) = \cancel{3xy^2} + 8\cancel{x^3y}$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = \text{const}$$

The solution is therefore

$$\phi(x,y) = \frac{3}{2}x^2y^2 + 2x^4y = C_{onst}$$

4 $\quad y'' + 9y = \sin 3t + e^{9t}$

a) homogeneous problem : $y'' + 9y = 0$

solution : $y = e^{rt}$: $r^2 + 9 = 0$

$r = \pm 3i$

solution $y_h(t) = c_1 \cos 3t + c_2 \sin(3t)$

b) To find the general solution, we need first a particular solution. One way to find this solution is by solving

(i) $y'' + 9y = \sin 3t$

(ii) $y'' + 9y = e^{9t}$ separately and next adding the solutions.

for (ii) we ~~first~~ guess a solution of the form $A e^{9t}$

$$81 A \cancel{e^{9t}} + 9 A \cancel{e^{9t}} = \cancel{e^{9t}}$$

$$90A = 1 \rightarrow A = \frac{1}{90}.$$

for (i) we first write the equation in complex form

$$z'' + 9z = e^{3it}$$

As e^{3it} is a solution of the homogenous problem we have to try a solution of the form:

$$z_p(t) = t A_1 e^{3it}$$

$$z_p' = A_1 e^{3it} + t(3iA_1) e^{3it}$$

$$z_p'' = A_1 t.(-9) e^{3it} + 6 i A_1 e^{3it}$$

$$z_p'' + 9z = \cancel{-9A_1 t e^{3it}} + 6iA_1 e^{3it} + \cancel{9tA_1 e^{3it}} = e^{3it}$$

$$A_1 = \frac{1}{6i} = \frac{-i}{6}$$

Hence we find the real solution $y(t) = \mathrm{Im}[t A_1 e^{3it}]$

$$= \mathrm{Im}\left[-\frac{it}{6}(\cos 3t + i\sin 3t)\right]$$

$$= -\frac{t}{6}\cos 3t$$

The general solution reads:

$$y(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{t}{6}\cos 3t + \frac{1}{90}e^{9t}$$

$$y(0) = 1 = c_1 + \frac{1}{90} \Rightarrow c_1 = \frac{89}{90}$$

$$y'(0) = 0 = 3c_2 - \frac{1}{6} + \frac{1}{10} = 0 \Rightarrow c_2 = \frac{2}{90}$$

so the solution in this case is $y(t) = \frac{89}{90}\cos 3t + \frac{2}{90}\sin 3t - \frac{t}{6}\cos 3t + \frac{e^{9t}}{90}$

5.

$$y'' + y = t^2 + 1 \qquad y(0) = y'(0) = 0$$

$$\mathcal{L}[y'' + y] = \mathcal{L}[t^2+1] = \frac{1}{s} + \frac{2}{s^3}$$

$$s^2\hat{y}(s) + \hat{y}(s) = \hat{y}(s)(s^2+1)$$

Hence $\hat{y}(s) = \frac{1}{(s^2+1)s} + \frac{2}{s^3(s^2+1)}$

$$= -\frac{s}{s^2+1} + \frac{1}{s} + 2\left[\frac{Ds+E}{s^2+1} + \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s}\right]$$

$$(Ds+E)s^3 + (A + Bs + Cs^2)(s^2+1) = 1$$

$\Rightarrow$ $D + C = 0$

$E + B = 0$

$A + C = 0$

$A = 1 \Rightarrow C = -1 \Rightarrow D = 1$

$B = 0 \Rightarrow E = 0$

$$\hat{y}(s) = \frac{1}{s} - \frac{s}{s^2+1} + \frac{2s}{s^2+1} + \frac{2}{s^3} - \frac{2}{s}$$

$$= \frac{s}{s^2+1} - \frac{1}{s} + \frac{2}{s^3}$$

It immediately follows that $y(t) = \cos t + t^2 - 1$

6. Gronwall's inequality

$$g(t) \leq C + Bt + k\int_0^t g(s)\,ds \qquad (*)$$

elegant way to prove $g(t) \leq Ce^{kt} + B\frac{e^{kt}-1}{k}$

define $u(t) = \int_0^t g(s)\,ds$

$$\frac{d\dot{u}}{dt} \leq C + Bt + k\,u(t)$$

$$\left(\frac{du}{dt} - ku\right) \leq C + Bt$$

multiply by e^{-kt}

$$\frac{d}{dt}(ue^{-kt}) = \left(\frac{du}{dt} - ku\right)e^{-kt} \leq Ce^{-kt} + Bte^{-kt}$$

integrate this equation from $\int_0^t$

$$u(t)\,e^{-kt} \leq \frac{Ce^{-kt}}{-k}\Bigg|_0^t + B\int_0^t s\,e^{-ks}\,ds$$

$$= \left(\frac{C - Ce^{-kt}}{k}\right) + \left(\frac{B - Be^{-kt}}{k^2}\right) - \frac{Bte^{-kt}}{k}$$

$$u(t) \leq \frac{Ce^{kt} - C}{k} + \frac{(Be^{kT} - B)}{k^2} - \frac{Bt}{k}$$

uit $(*)$ volgt: $\frac{g(t) - C - Bt}{k} \leq \int_0^t g(s)\,ds = u(t)$

thus $\frac{g(t) - \cancel{C} - \cancel{Bt}}{\cancel{k}} \leq \frac{Ce^{kt}}{\cancel{k}} - \cancel{\frac{C}{k}} + \frac{B(e^{kt}-1)}{k^2} - \cancel{\frac{Bt}{k}}$

therefore $g(t) \leq Ce^{kt} + \frac{B(e^{kt}-1)}{k} \quad \forall t \in [0,T]$