Exam Ordinary Differential Equations, TW2030
Monday 28 January 2019, 13.30-16.30h

- This exam consists of 5 problems.
- All answers need to be justified.
- Norm: total of 46 points; the distribution of points is a shown in the exercises. The exam grade is (total points+4)/5.
- *Success* !

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1. The following differential equation for the function $x : \mathbf{R} \to \mathbf{R}$ is given

$$\frac{dx}{dt} = x^2.$$

a. (3) Find the solution $x(t)$ using initial condition $x(0) = 1$, and give the maximal interval of existence of the solution.

The differential equation for the function $y : \mathbf{R} \to [1, \infty)$ reads

$$\frac{dy}{dt} = \sqrt{y-1}.$$

b. (3) Find a solution $y(t)$ using initial condition $y(0) = 1$, and argue whether this solution is unique or not.

2. Consider the initial value problem for the function $y = y(x)$:

$$x^2 y'' + 2xy' - \alpha^2 x^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 0. \tag{1}$$

where $\alpha \in \mathbf{R}$ and $\alpha \neq 0$.

a. (1) Find a singular point of (1) and determine whether it is regular or irregular.

b. (2) Give the indicial equation and show that $r = 0$ is a solution; what is the other solution?

c. (3) Find two linearly independent solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ by giving the recurrence relations that a_n satisfies. Mention for which values of n each of the recurrence relations holds.

d. (2) Solve the recurrence relations obtained in (c) and give the solution of the initial value problem.

3. We consider the matrix differential equation for function $\mathbf{x}(t) \in \mathbb{R}^3$:

$$\dot{\mathbf{x}} = A\,\mathbf{x},$$

with matrix A defined by

$$A = \begin{pmatrix} -1 & a & 0 \\ 0 & -1 & a \\ 0 & 0 & -1 \end{pmatrix},$$

where $a \in \mathbf{R}$ is a constant.

a. (7) Calculate e^{At} and give the solution of $\dot{\mathbf{x}} = A\,\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$.

b. (4) Determine a particular solution of the nonhomogeneous problem

$$\dot{\mathbf{x}} = A\,\mathbf{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t \sin t.$$

4. The following system of differential equations is given:

$$\begin{aligned} \dot{x} &= (x^2 - 1)y, \\ \dot{y} &= (x + 2)(y - 1)(y + 2). \end{aligned} \tag{2}$$

a. (2) Calculate all equilibria.

b. (2) Characterise all equilibria (saddle, center, node, spiral/focus) and determine (if possible) the stability (asymptotically stable, stable, unstable) of each of the equilibria for the *linearised system* that is obtained after linearization around an equilibrium.

c. (2) Determine (if possible) the stability of the equilibria for the *non-linear* system (2) and motivate your answer.

d. (3) Give an expression for the orbits.

e. (3) Show that there are infinitely many periodic solutions whose orbits enclose $(-2, 0)$.

f. (3) Sketch the phase portrait. Draw the equilibria, periodic trajectories, some (special) orbits, and indicate the flow of the solutions by arrows.

5. (6) Show that if $X(t) \in \mathbf{R}^n \times \mathbf{R}^n$ is a fundamental solution of

$$\dot{\mathbf{x}} = A(t)\mathbf{x},$$

with $A(t) \in \mathbf{R}^n \times \mathbf{R}^n$ then $Y(t) = \left(X^T(t)\right)^{-1}$ is a fundamental solution of

$$\dot{\mathbf{y}} = -A^T(t)\mathbf{y},$$

where the superscript ^{T} denotes transposition.