**Delft University of Technology** **Faculty EEMCS**

**Mid term Exam Ordinary Differential Equations, AM2030**
**Tuesday 10 December 2019, 13.30-15.30h**

- This exam consists of 5 regular problems and one bonus problem.
- All answers need to be justified.
- Norm: total of 35 points; the distribution of points is as shown in the exercises. The exam grade is (total points)/3.5.
- *Success* !

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1. The following differential equation for the function $x : \mathbb{R} \to \mathbb{R}$ is given

$$\frac{dx}{dt} = x^2 + 1.$$

a. (3) Find the solution $x(t)$ using initial condition $x(0) = 1$, and give the maximal interval of existence of the solution.

The differential equation for the function $y : \mathbb{R} \to [1, \infty)$ reads

$$\frac{dy}{dt} = t^2 \sqrt{y - 1}.$$

b. (3) Find a solution $y(t)$ using initial condition $y(0) = 1$, and argue whether this solution is unique or not.

2. The differential equation for the function $y = y(x)$ is

$$(x-1)^2 y'' + 2x(x-1)y' + (x-1)y = 0, \quad x > 1. \tag{1}$$

a. (1) Show that $x = 1$ is a regular singular point of equation (1).

b. (4) Give the indicial equation $F(r) = 0$ and find the two solutions for $r$ and write the recurrence relation for the coefficients $a_n$ ( **You do not need to solve it!**)

c. (2) Determine the first 3 nonzero terms of the series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^{n+r},$$

only for the **largest root** of the indicial equation found in (b).

d. (2) What can be said about the radius of convergence of this series?

e. (2) How would you determine a second solution of (1) linearly independent solution of the one found in c)?
**You do not need to do the calculation, giving the method is sufficient!**

3. Consider the following differential equation for $y(x)$:

$$(3y^2 + 8x^2y) + (3xy + 2x^3)\frac{dy}{dx} = 0$$

a. (2) Is the differential equation exact?

b. (4) Determine the general solution of this differential equation. If necessary find an integrating factor.

4. The following differential equation for $y(t)$ is given:

$$y'' + 9y = \sin(3t) + e^{9t} \tag{2}$$

a. (2) Find the solution of the homogeneous problem.

b. (2) Find the general solution of (2).

c. (2) Assume that the initial conditions are $y(0) = 1, y'(0) = 0$, give the solution of the initial value problem.

5. (6) A differential equation for the function $y(t)$ with initial conditions is given as

$$y'' + y = t^2 + 1, \qquad y(0) = 0, y'(0) = 0. \tag{3}$$

Find the solution of this differential equation **using the Laplace transform!**

**BONUS:** *Generalization of Gronwall's inequality*

6. (3.5) Show that if $g : [0, T] \to \mathbb{R}$ is continuous and if there exist nonnegative constants $C, B, K$, such that

$$g(t) \le C + Bt + K \int_0^t g(s)\, ds, \quad 0 \le t \le T$$

then

$$g(t) \le Ce^{Kt} + B\frac{e^{Kt} - 1}{K} \quad 0 \le t \le T$$

*Hint: You can use the same techniques as we used to prove uniqueness of solutions.*