

Key points of attention:

- 1) This exam will be an open book exam. This means that you are only allowed to use your book, lecture notes, and the homework exercises you made.
- 2) You will have at most 15 minutes after the exam for uploading your exam in one (readable) pdf file in Brightspace/assignment/EXAM AM2070 June 2020. So, your exam has to be handed in before 16.45 h.
- 3) This exam will be a pass/fail exam. When your work has been graded, a number of students will be randomly chosen to have a remote check of 15 minutes.
- 4) On the exam you have to write down your name and student ID number, and you have to write down and to undersign the following statement: I declare that I have made this examination on my own, with no assistance, and in accordance with the TU Delft policies on plagiarism, cheating, and fraud.
- 5) This exam consists of 4 exercises. For each exercise 10 points in total can be obtained as indicated in the exercise. Exam grading = number of points divided by 4. The final grading for this course is in line with the earlier determined rules (see also earlier announcements on Brightspace).

Exercise 1:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x, t)$:

$$\begin{aligned} u_t &= u_{xx} + Q(x, t), & 0 < x < L, t > 0, \\ Au_x(0, t) &= u(0, t), & t \geq 0, \\ u_x(L, t) &= -Bu(L, t), & t \geq 0, \\ u(x, 0) &= f(x), & 0 < x < L, \end{aligned}$$

where $Q(x, t)$ and $f(x)$ are sufficiently smooth functions, and where $L > 0$, $A \geq 0$, and $B \geq 0$ are constants.

- 1pt a) Give a physical interpretation of the initial-boundary value problem.
- 2pt b) Assume that $Q(x, t) = e^{-2x}$. Determine the equilibrium solution (that is, determine u for $t \rightarrow \infty$) for those values of A and B for which an equilibrium solution exists.
- 2pt c) Take $A = B = 0$, and $Q(x, t) \equiv 0$. Determine $u(x, t)$.
- 2pt d) Describe how the solution of the initial-boundary value problem can be determined for arbitrary A , B , $f(x)$, and $Q(x, t)$. The solution does not have to be computed, but clearly indicate how the solution can be computed (that is, give a description of the method and the steps that have to be made).
- 3pt e) Prove that the initial-boundary value problem has at most one solution (that is, give a uniqueness proof).

Exercise 2:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x, t)$:

$$\begin{aligned} u_{tt} &= u_{xx}, & x > 0, t > 0, \\ au_x(0, t) &= bu(0, t) - h(t), & t \geq 0, \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), & x > 0, \end{aligned}$$

where h, f , and g are sufficiently smooth functions, and in which a and b are non-negative constants.

- 1pt a) Give a physical interpretation of the problem.
- 3pt b) Take $a = b = 1$, $f \equiv 0$, and $g \equiv 0$. The general solution of the partial differential equation is given by: $u(x, t) = F(x - t) + G(x + t)$. Determine the solution of the initial-boundary value problem by using the general solution of the PDE.
- 3pt c) Take $a = 1$, $b = 0$, $h \equiv 0$, and $f \equiv 0$. First extend the problem to a problem on $-\infty < x < \infty$. Then, determine the function $u(x, t)$ by using the Fourier-transform method.
- 3pt d) Take $a = 0$, $b = 1$, $f(x) \equiv 0$ and $g \equiv 0$. Determine $u(x, t)$ by using the Laplace-transform method.

Exercise 3:

Consider the following boundary value problem for the twice continuously differentiable function $u(x, y, z)$:

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= Q(x, y, z), & x > 0, y > 0, z > 0, \\ u_z(x, y, 0) &= 0, & x > 0, y > 0, \\ u(0, y, z) &= f(y, z), & y > 0, z > 0, \\ u(x, 0, z) &= 0, & x > 0, z > 0, \end{aligned}$$

where Q , and f are sufficiently smooth functions, and where u , Q , and f (as well as their derivatives) tend to zero for $|\underline{x}| \rightarrow \infty$.

- 1pt a) Give a physical interpretation of the problem.
- 3pt b) The Green's function for the Laplace operator in $\mathbb{R}^3$ is given by

$$G(\underline{x}; \underline{x}_0) = \frac{-1}{4\pi |\underline{x} - \underline{x}_0|}.$$

Determine the Green's function for the given boundary value problem.

- 3pt c) Determine $u(x, y, z)$.
- 3pt d) Prove that the solution of the boundary value problem is unique.

Exercise 4:

Consider the following initial value problem for the continuously differentiable function $u(x, t)$:

$$\begin{aligned} u_t + (1 - 2u)u_x &= 0, & -\infty < x < \infty, \quad t > 0, \\ u(x, 0) &= f(x), & -\infty < x < \infty, \end{aligned}$$

where $f(x)$ is a given function.

- 1pt a) Give a (physical) interpretation of the problem.
- 4pt b) Determine $u(x, t)$, and make clear in different figures how the solution evolves in time, when

$$f(x) = \begin{cases} 1 & \text{for } x < 0, \\ \frac{3}{4} & \text{for } 0 < x < 1, \\ \frac{1}{4} & \text{for } x > 1. \end{cases}$$

- 5pt c) Determine $u(x, t)$, and make clear in different figures how the solution evolves in time, when

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x < 0, \\ \frac{3}{4} & \text{for } 0 < x < 1, \\ 1 & \text{for } x > 1. \end{cases}$$