Delft University of Technology
Faculty EEMCS
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Exam part 2 Real Analysis (AM2090)
22-1-2021; 9.00–11.00
Teacher M.C. Veraar, co-teacher K.P. Hart.

You can use the lecture notes by Mark Veraar, the answers to exercises which I provided and your own notes and answers. Do not contact others to help you with your exam. Scan your work and put it into one pdf file. It is your responsibility that the work is well-readable. Hand in via Brightspace assignments. The deadline for handing in is 11.30. The extra time is meant for solving unexpected practical situations. If you have right to extra time, then you can hand in at 12.00. Please send "the extra time form" by email to m.c.veraar@tudelft.nl if you have not done this yet.

You can cite theorem/exercise numbers from the book and lecture notes. Suggested abbreviations: Thm x.y for theorem x.y from the lecture notes by Mark Veraar.

Please write this line at the beginning of the exam you hand in and sign it:
"**I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud**".

(12) 1. Let $(S, \mathcal{A}, \mu)$ be a measure space. Let $A_n \in \mathcal{A}$ for each $n \geq 1$. Show that $\mu\Big(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n\Big) \leq \lim_{k\to\infty} \inf_{n\geq k} \mu(A_n)$.

(6) 2. a. Give an example of a sequence of *continuous* functions from $\mathbb{R}$ to $[0, \infty)$ that converges pointwise to 0 and such that $\int_{\mathbb{R}} f_n d\lambda \not\to 0$, where λ denotes the Lebesgue measure. As always provide proofs of your assertions.

(12) b. Let $(S, \mathcal{A}, \mu)$ be a measure space and $(f_n)_{n\geq 1}$ a sequence of integrable functions with value in $[0, \infty)$ that converges pointwise to an integrable function $f : S \to [0, \infty)$. Prove

$$\int_S f_n d\mu - \int_S f d\mu - \|f_n - f\|_1 \to 0.$$

Hint: Apply the Dominated Convergence Theorem to $\min\{f_n, f\}$ and use $x \wedge y = \frac{1}{2}(x+y) - \frac{1}{2}|x-y|$.

3. Let λ denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \in (0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

(4) a. Show that f is measurable via Proposition 4.5(iii).

(8) b. Show that f is integrable.
Hint: First calculate $\int_{[t,1]} f d\lambda$ with $t \in (0, 1)$ by using Example 5.16 from the lecture notes and the Fundamental Theorem of Calculus.

Let $(q_n)_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - q_n)$.

(8) c. Show that g is integrable.
Hint: Use Exercise 5.10 and Corollary 6.8 from the lecture notes.

See also the next page.

(12) 4. Let $n \geq 0$ and let $f = \sum_{|k| \leq n} c_k e_k$ be a trigonometric polynomial. Show that $\|f'\|_2 \leq n\|f - \widehat{f}(0)\|_2$.
Hint: Use $e_k' = ike_k$.

(11) 5. Let S be a set. For subsets $A, B \subseteq S$ define $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Suppose $\mathcal{R} \subseteq \mathcal{P}(S)$ is nonempty, and that the following properties hold:

$$\text{(i)} \quad A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}, \qquad \text{and} \qquad \text{(ii)} \quad A, B \in \mathcal{R} \Rightarrow A \triangle B \in \mathcal{R}.$$

Show that $\mathcal{R}$ is a ring.
Hint: Of course you should use Remark 1.2(1).

6. Let λ denote the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Let $D = \{(x, x) \in \mathbb{R}^2 : x \in [0, 1]\}$.

(3) a. Explain why D is a closed set, and conclude that $D \in \mathcal{B}(\mathbb{R}^2)$.

Let $x_j = j/n$ for $j \in \{-1, 0, 1, \ldots, n\}$.

(4) b. Show that $D \subseteq \bigcup_{j=0}^{n} (x_{j-1}, x_j] \times (x_{j-1}, x_j]$.

(10) c. Use (b), and properties of measures to show that $\lambda(D) = 0$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END