Exam Relaxations and Heuristics (WI4515)

19 January 2021, 09.00–12.00 (3 hours).

The exam consists of 5 questions worth 10 points each. Your grade is given by $1 + \frac{9p}{50}$, where p is the total number of points obtained. You are **not allowed** to consult anyone else or to use the internet. The total number of pages of this exam is 3. **Good luck!**

Please **write down** the following statement on your solution sheet: "I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud. I promise that I will not use unauthorized help from people or other sources during my exam. I will create the answers on my own and I will create them only during the allocated exam time slots."

The exam should be **handwritten**, either on paper or digital. Please **number** all your pages and start each exercise on a **new page**. Please submit only **one pdf** file. Include a copy of your **student ID**.

Let A and B be sets such that $B \subseteq A$. Recall that the *characteristic vector* of B is the vector $\chi_B \in \mathbb{R}^A$ (that is, a function $\chi_B\colon A \to \mathbb{R}$) such that $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ otherwise.

1. Let $G = (V, E)$ be a graph. Consider the maxcut polytope:

$$P(G) = \text{conv}\{\, \chi_S \in \mathbb{R}^E : S \subseteq \delta(U) \text{ for some } U \subseteq V \,\},$$

that is, $P(G)$ is the convex hull of all characteristic vectors of sets S of edges that are contained in some cut $\delta(U)$.

[7pts] (a) Give an integer programming formulation for $P(G)$ having polynomially many (on $|V|$) constraints and variables. *N.B.:* You need to give a *linear* formulation, so no products of variables!

[3pts] (b) What is the strength of your formulation? If you optimize over the linear programming relaxation, how far can you get from the integer optimal?

2. For each of the three sets below, find a missing valid inequality and verify graphically that its addition to the formulation gives $\text{conv}(X)$.

[3pts] (a) $X = \{\, (x, y) \in \mathbb{R}^1_+ \times \{0,1\}^1 : x - 12y \leq 1,\ x \leq 5\}$.

[3pts] (b) $X = \{\, x \in \{0,1\}^2 : -5x_1 + 2x_2 \leq 1\}$.

[4pts] (c) $X = \{\, (x, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1_+ : x \leq 3y, x \leq 10\}$.

3. Consider again the maxcut polytope of Question 1.

[3pts] (a) What is the dimension of $P(G)$?

[7pts] (b) Say that $T \subseteq E$ is a set of edges forming a triangle in G. Show that the inequality

$$\sum_{e \in T} x_e \leq 2$$

induces a facet of $P(G)$. *N.B.:* You need to show that this inequality is valid and that it induces a facet!

[10pts] 4. Consider a hospital setting with a set of nursing wards J where patients from a set of medical specialties I need to be admitted. The problem is to assign each specialty to one or more nursing wards such that enough beds are available and such that the number of wards each specialty is assigned to is minimized. The total number of beds needed by medical specialty $i \in I$ is given by D_i and the number of beds available on nursing ward $j \in J$ is given by C_j.

To model this, we introduce decision variables x_{ij} which take the fraction of beds used by medical specialty $i \in I$ on nursing ward $j \in J$. In addition, we introduce binary decision variables y_{ij}, which are one when medical specialty $i \in I$ is using one or more beds on nursing ward $j \in J$.

Now, the problem can be formulated as:

$$\min \sum_{i \in I} \sum_{j \in J} y_{ij}$$

$$\begin{aligned}
\sum_{j \in J} x_{ij} &= 1, && \forall i \in I,\\
\sum_{i \in I} x_{ij} D_i &\leq C_j, && \forall j \in J,\\
x_{ij} &\leq y_{ij}, && \forall i \in I, j \in J,\\
x_{ij} &\geq 0, && \forall i \in I, j \in J,\\
y_{ij} &\in \{0, 1\}, && \forall i \in I, j \in J.
\end{aligned}$$

Formulate the master and subproblems for this problem.

5. In the multiple knapsack problem, we are given a set N of items, a set $M = \{1, \ldots, k\}$ of knapsacks, a *weight function* $f\colon N \to \mathbb{R}_+$ assigning a weight to each item, a *capacity function* $w\colon M \to \mathbb{R}_+$ assigning a capacity to each knapsack, and a cost function $c\colon N \times M \to \mathbb{R}_+$ which tells us the cost of assigning item $i \in N$ to knapsack $k \in M$. Our goal is to decide which items to pick and in which knapsack each picked item goes so as to maximize the total cost while satisfying the capacity of each knapsack.

More precisely, a feasible solution of the multiple knapsack problem is a k-tuple $(S_1, \ldots, S_k)$, where S_1, …, S_k are pairwise-disjoint subsets of items such that $f(S_j) \leq w_j$ for all $j = 1$, …, k. The cost of the solution is

$$\sum_{j=1}^{k} \sum_{i \in S_j} c(i, j).$$

To each feasible solution $\mathcal{S} = (S_1, \ldots, S_k)$ of the knapsack problem we assign a vector $x_{\mathcal{S}}\colon N \times M \to \{0, 1\}$ by setting $x_{\mathcal{S}}(i, j) = 1$ if and only if $i \in S_j$.

[3pts] (a) Let P be the multiple knapsack problem polytope, defined as the convex hull of all vectors $x_{\mathcal{S}}$ for each feasible solution $\mathcal{S}$ of the multiple knapsack problem. Give an integer programming formulation for P.

[3pts] (b) A *multicover* is a pair (S, J) with $S \subseteq N$ and $J \subseteq M$ such that $f(S) > w(J)$. Show that if (S, J) is a multicover, then

$$\sum_{i \in S} \sum_{j \in J} x(i, j) \leq |S| - 1$$

is a valid inequality for P. This is the *multicover inequality*.

[4pts] (c) A multicover (S, J) is *minimal* if for every $i \in S$ there is a valid assignment of the items in $S \setminus \{i\}$ to the knapsacks in J (that is, an assignment that satisfies the knapsack capacities). Show that, if (S, J) is *not* minimal, then the multicover inequality is *not* a facet of P.

End of test