Technische Universiteit Delft — Mekelweg 4
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Exam Statistical Inference (WI4455)
January 21, 2020, 9.00–12.00

- This is an open book exam to be made by you without consulting others.
- Unless stated differently, always add an explanation to your answer.
- Scan your work and put it into *one pdf-file*. It is your responsibility that the work is well readable.
- Try to have the size not too large (20 Mb can lead to delays). The name of your file should be your netid with study number (example: `netid1234567.pdf`).
- The exam is available on January 20 at 9.00 and ends at 12.00. You have to upload your work before 12.30 in Brightspace (so definitively *not* by sending an email to me).
- If you are allowed extra time, then the exam ends at 12:30 and you have to upload your scan before 13:00. Please send the "extra time form" to f.h.vandermeulen@tudelft.nl if you have not already done so.
- Write your name and study number on every page.
- Write the following line at the top of page 1 of the paper on which you write your solutions and sign it:

  I promise that I have not used unauthorized help from people or other sources for completing my exam. I created the submitted answers all by myself during the exam.
  Date:
  Name:
  Signature:

- In case of questions about the exam, or technical problems at an earlier stage, send me an email at f.h.vandermeulen@tudelft.nl; I'll be monitoring my inbox regularly the entire duration of the exam.
- The exam is not the same for all students, depending on the digits of your student-number you'll make a particular version of the exam. Each student will make four exercises.
  - If the first digit of your student-number is in the set $\{1, 2, 3, 4, 5\}$, you make exercise 1A, else you make exercise 1B.
  - If the second digit of your student-number is in the set $\{1, 2, 3, 4, 5\}$, you make exercise 2A, else you make exercise 2B.
  - If the third digit of your student-number is in the set $\{1, 2, 3, 4, 5\}$, you make exercise 3A, else you make exercise 3B.

– If the fourth digit of your student-number is in the set $\{1, 2, 3, 4, 5\}$, you make exercise 4A, else you make exercise 4B.

Please be careful with making the correct exercises, making the wrong version will lead to subtraction of points!

1A Assume for for $i \in \{1, \dots, n\}$

$$X_i = \begin{cases} 0 & \text{with probability } 1-\theta \\ -1 & \text{with probability } \theta/3 \\ 1 & \text{with probability } 2\theta/3 \end{cases},$$

where $\theta \in (0, 1)$. Let $S_j = \sum_{i=1}^n \mathbf{1}_{\{X_i = j\}}$, where $j \in \{-1, 0, 1\}$. Assume $S_0 > 0$.

(a) [1 pt]. Find a one-dimensional sufficient statistic for θ.

(b) [2 pt].Compute a maximum likelihood estimator for θ.

(c) [2 pt]. Suppose we take the Bayesian point of view, where $X_1, \dots, X_n \mid \Theta = \theta$ has the given distribution, and additionally a prior on Θ is used. Using Bayesian notation, the prior density satisfies $p(\theta) \propto \theta(1-\theta)^2$. Derive the posterior distribution of θ.

(d) [2 pt]. With the given prior, we consider the Bayes rule using loss-function $L(\theta, a) = |\theta - a|$. Is the Bayes rule admissible? Argue why/why not.

1B We say that the random variable X has the Shifted Geometric distribution with parameter $\theta \in (0, 1)$, denoted by $X \sim \text{SGeo}(\theta)$, if it has probability mass function

$$f_X(x; \theta) = \text{P}_\theta(X = x) = (1-\theta)^x \theta, \qquad x = 0, 1, \dots .$$

We have $\text{E}_\theta X = (1-\theta)/\theta$. Suppose $X_1, \dots, X_n$ are independent and identically distributed with $X_i \sim \text{SGeo}(\theta)$.

(a) [1 pt]. Give an unbiased estimator for θ. That is, state the estimator and prove it is unbiased for θ.

(b) [3 pt]. Derive the maximum likelihood estimator for $\tau = \theta/(1-\theta)$.

(c) [3 pt]. Suppose $n = 1$ and we wish to test the hypothesis $H_0 : \theta = 1/4$ versus $H_1 : \theta = \theta_1$ with $\theta_1 > 1/4$. In case we obtain the realisation $x = 2$, compute the p-value of the test with X_1 as test statistic.

2A Let $\theta \in (0, \infty)$ be an unknown parameter and X be a random variable such that $\mathbb{E}[X] = c\theta$ and $\text{Var}(X) = \nu(\theta)$, where $\nu(\theta)$ is specified and $c \in \mathbb{R}$ fixed. Consider estimation of θ by linear functions of the form $d_a(X) = aX$ for $a \in (0, 1)$, with squared-error loss. Let $\mathcal{A}$ be the set of all such estimators, indexed by $a \in (0, 1)$.

(a) [3 pt]. For $\nu(\theta) = \theta^2$, calculate the risk function of d_a, and show that for $c = 2$ the rule $d_{1/2}$ is inadmissible in the class $\mathcal{A}$.

(b) [3 pt]. Now fix $c = 1$. For $\nu(\theta) = \theta$, prove that every member of $\mathcal{A}$ is admissible among the class $\mathcal{A}$.

2B Let $\theta \in (0, \infty)$ be an unknown parameter and X be a random variable such that $\mathbb{E}[X \mid \Theta = \theta] = c\theta$ and $\mathrm{Var}(X \mid \Theta = \theta) = \nu(\theta)$, where $\nu(\theta)$ is specified and $c \in \mathbb{R}$ fixed. Consider estimation of θ by linear functions of the form $d_a(X) = aX$ for $a \in (0, 1)$, with squared-error loss. Let $\mathcal{A}$ be the set of all such estimators, indexed by $a \in (0, 1)$.

(a) [3 pt]. For $\nu(\theta) = \theta^2$, calculate the risk function of d_a, and show that for $c = 2$ the rule $d_{1/2}$ is inadmissible in the class $\mathcal{A}$.

(b) [3 pt]. Suppose $\nu(\theta) = \theta^k$ where k is a positive integer. Find a closed form expression for the Bayes estimator in the class $\mathcal{A}$ when θ has prior density $f_\Theta(\theta) = e^{-\theta}\mathbf{1}_{[0,\infty)}(\theta)$. You may use that for integers n we have $\int_0^\infty x^n e^{-x} \mathrm{d}\, x = n!$.

3A Suppose X has density

$$f_X(x; \theta) = \frac{\theta}{x^{\theta+1}} \mathbf{1}_{[1,\infty)}(x).$$

Assume $\theta > 1$. Then $\mathrm{E}_\theta\, X = \theta/(\theta - 1)$.

Suppose $\theta_1 > \theta_0 > 0$ and we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Define the decision rules $d_c(X)$ by

$$d_c(X) = \begin{cases} a_0 = \{\text{accept } H_0\} & \text{if } X > c \\ a_1 = \{\text{accept } H_1\} & \text{if } X \leq c \end{cases},$$

where $c > 1$.

(a) [2 pt]. Incorrectly accepting H_0 is considered two times as costly as incorrectly accepting H_1. Write down a loss function that reflects this.

(b) [3 pt]. Using this loss function, derive an expression for the risk-function of the rule $d_c(X)$, both for $\theta = \theta_0$ and for $\theta = \theta_1$. Your answer should only depend on θ_0, θ_1 and c.

(c) [3 pt]. Suppose $\theta_0 = 2$ and $\theta_1 = 4$. Compute the minimax rule. That is determine c such that d_c is minimax.

3B Suppose X has density

$$f_X(x; \theta) = \frac{\theta}{x^{\theta+1}} \mathbf{1}_{[1,\infty)}(x).$$

Assume $\theta > 1$. Then $\mathrm{E}_\theta\, X = \theta/(\theta - 1)$.

Suppose $\theta_1 > \theta_0 > 0$ and we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Define the decision rules $d_c(X)$ by

$$d_c(X) = \begin{cases} a_0 = \{\text{accept } H_0\} & \text{if } X > c \\ a_1 = \{\text{accept } H_1\} & \text{if } X \leq c \end{cases},$$

where $c > 1$.

(a) [2 pt]. Incorrectly rejecting H_1 is considered three times as costly as incorrectly rejecting H_0. Write down a loss function that reflects this.

(b) [3 pt]. Using this loss function, derive an expression for the risk-function of the rule $d_c(X)$, both for $\theta = \theta_0$ and for $\theta = \theta_1$. Your answer should only depend on θ_0, θ_1 and c.

(c) [3 pt]. Suppose $\theta_0 = 2$ and $\theta_1 = 4$. If a priori H_0 is considered twice as likely as H_1, derive the Bayes decision rule.

4A In this exercise Bayesian notation is used; in particular, all random quantities are denoted with lower-case letters. Assume $1 \leq i \leq n$ and $1 \leq j \leq M$ are indices and that we observe y_{ij}. It is assumed that

$$y_{ij} \mid \alpha_1, \ldots, \alpha_M \overset{\text{ind}}{\sim} N\left(\alpha_i, \sigma^2\right)$$
$$\alpha_1, \ldots, \alpha_M \overset{\text{iid}}{\sim} N\left(0, 1/\tau\right)$$

Here, τ is considered as hyperparameter.

(a) [2 pt]. Define $\alpha = [\alpha_1, \ldots, \alpha_M]$. By stacking all y_{ij} into a vector y, show that we have

$$y \mid \alpha \sim N\left(X\alpha, \sigma^2 I_n\right).$$

(b) [3 pt]. Show that the posterior of α is a multivariate normal distribution. Derive its parameters.

(c) [3 pt]. In this model we have $y \sim N(\mu, \Sigma)$. Specify μ and Σ and explain how this result can be used to determine τ with empirical Bayes.

4B In this exercise Bayesian notation is used; in particular, all random quantities are denoted with lower-case letters. Assume $1 \leq i \leq n$ and $1 \leq j \leq M$ are indices and that we observe y_{ij}. It is assumed that

$$y_{ij} \mid \alpha_1, \ldots, \alpha_M \overset{\text{ind}}{\sim} N\left(\alpha_i, \sigma^2\right)$$
$$\alpha_1, \ldots, \alpha_M \overset{\text{iid}}{\sim} N\left(0, 1/\tau\right)$$

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$$y \mid \alpha \sim N\left(X\alpha, \sigma^2 I_n\right).$$

(b) [3 pt]. Show that the posterior of α is a multivariate normal distribution. Derive its parameters.

(c) [3 pt]. As we are uncertain about τ, we add a layer to the hierarchical model, where τ get the $Exp\,(1)$-distribution. Give the steps of a Gibbs sampler for drawing from the posterior of (α, τ).