DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

The final grade of the test: (total number of points)/5

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TEST SCIENTIFIC COMPUTING (wi4201 / wi4201COSSE)
Wednesday January 19 2022, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. $\kappa_2(A) = 1$, where κ_2 is the 2-norm condition number. (2 pt.)

   (b) the amount of work per iteration of the Bi-CGSTAB method remains constant as a function of the number of iterations; (2 pt.)

   (c) Given matrix $A \in \mathbb{R}^{n\times n}$. $P \in \mathbb{R}^{n\times n}$ is a permutation matrix. The spectra of A and $P^T AP$ are the same. (2 pt.)

   (d) $A \in \mathbb{R}^{n\times n}$ is such that $a_{i,i} = 4, a_{i,i-1} = -1, a_{i-1,i} = -1$. All other components of A are equal to zero. $\|A\|_2 = 8$. (2 pt.)

   (e) $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ is invertible. (2 pt.)

2. Given the domain $\Omega = (0,1)\times(0,1)$ with boundary $\Gamma = \partial\Omega$, we consider the following problem:

$$-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} + ku(x,y) = f(x,y)$$

   where $k > 0$, supplied with Dirichlet boundary conditions

$$u(x,y) = 1 \text{ on } \Gamma$$

   For the discretization we use an equidistant grid, step size h, and lexicographic ordering of the unkowns. Please answer the following questions:

   (a) Check that the function given by:

$$u^{[k\ell]}(x,y) = \sin(k\pi x)\sin(\ell\pi y) \text{ for } k,\ell \in \mathbb{N}, k \neq 0 \text{ and } \ell \neq 0$$

   is an eigenfunction of operator $-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} + ku(x,y)$, and give an expression for the eigenvalue.
   (Hint: $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$). (2.5 pt.)

(b) For an internal grid point the stencil is given by:

$$\frac{1}{h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4+kh^2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Show that the numerical method has a local truncation error of $O(h^2)$. (2.5 pt.)

(c) Give the stencil and right-hand side for the grid point located at $(x, y) = (1-h, 1-h)$. (2.5 pt.)

(d) Show that the corresponding coefficient matrix A is positive definite. (2.5 pt.)

3. (a) Given the linear system $A\mathbf{u} = \mathbf{f}$ with $A \in \mathbb{R}^{n\times n}$. Consider a splitting of the form $A = M - N$ where M is non-singular. Derive a recursion formula for the BIM (Basic Iterative Method) iterates $\mathbf{u}^k$. Derive a recursion formula for the residual vector $\mathbf{r}^k$. (2 pt.)

(b) Give the iteration matrix B for this BIM and give a sufficient condition such that the BIM converges. (2 pt.)

(c) Suppose A is a lower triangular matrix. Show that the Gauss Seidel method converges for such a matrix. (2 pt.)

(d) Suppose A is a lower triangular matrix. Show that the Jacobi method converges for such a matrix. (2 pt.)

(e) Give three different stopping criteria and specify the good and bad properties of these stopping criteria. (2 pt.)

4. Consider the linear system $A\mathbf{u} = \mathbf{f}$, where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix.

(a) If A is SPD show that $< \mathbf{y}, \mathbf{z} >_A = \mathbf{y}^T A\mathbf{z}$ is an inner product.
(Hint: an inner product has the following properties: $< \mathbf{u}, \mathbf{v} > = < \mathbf{v}, \mathbf{u} >$, $< c\,\mathbf{u}, \mathbf{v} > = c < \mathbf{u}, \mathbf{v} >$, $< \mathbf{u} + \mathbf{v}, \mathbf{w} > = < \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >$ and $< \mathbf{u}, \mathbf{u} > \geq 0$ with equality only for $\mathbf{u} = 0$) (2 pt.)

(b) We assume that $\mathbf{u}^1 = \alpha_0 \mathbf{r}^0$. Determine α_0 such that $\|\mathbf{u} - \mathbf{u}^1\|_A$ is minimal. (2 pt.)

(c) The matrix A corresponds to a shifted discretized Poisson operator. The eigenvalues are given by

$$\lambda_{k,\ell} = 6 - 2\cos\frac{\pi k}{61} - 2\cos\frac{\pi \ell}{61} \quad , \quad 1 \leq k, \ell \leq 60.$$

Determine the linear rate of convergence for the Conjugate Gradient method. (2 pt.)

(d) If the convergence of the Conjugate Gradient method is too slow a preconditioner M could be used. Give three properties for matrix M in orde to be a suitable preconditioner. How can such a preconditioner be combined with Conjugate Gradient in order to obtain the Preconditioned Conjugate Gradient method? (2 pt.)

(e) Consider matrix A given by: $A = \begin{pmatrix} 100 & -1 & 0 \\ -1 & 100 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Give an estimate of the convergence of CG. Take for the preconditioner a diagonal matrix and give an estimate of the convergence of PCG. (2 pt.)

5. Consider the Power method to approximate the eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$. The eigenvalues are ordered such that $|\lambda_1| > |\lambda_2| \geq ... \geq |\lambda_n|$ and $\lambda_1 \in \mathbb{R}$.

   (a) The basic Power method is given by: $\mathbf{q}_k = A\mathbf{q}_{k-1}$. We assume that $\mathbf{q}_0$ can be written as a linear combination of the eigenvectors, with a non-zero component of the eigenvector corresponding to λ_1. Define $\lambda^{(k)} = \frac{\mathbf{q}_k^T A \mathbf{q}_k}{\|\mathbf{q}_k\|_2^2}$ and show that

   $$|\lambda_1 - \lambda^{(k)}| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) .$$

   (2.5 pt.)

   (b) Now we consider the advanced Power method:
   $\mathbf{q}_0 \in \mathbb{R}^n$ is given
   for $k = 1, 2, ...$
   $\mathbf{z}_k = A\mathbf{q}_{k-1}$
   $\mathbf{q}_k = \mathbf{z}_k / \|\mathbf{z}_k\|_2$
   $\lambda^{(k)} = \bar{\mathbf{q}}_{k-1}^T \mathbf{z}_k$
   endfor
   Show that if $\mathbf{q}_k$ is close to the eigenvector corresponding to λ_1 then $\lambda^{(k)}$ is a good approximation of λ_1. (2.5 pt.)

   (c) Note that from part (a) it follows that the Power method is a linearly converging method. Give a stopping criterion for the Power method. (2.5 pt.)

   (d) Given a matrix $A \in \mathbb{R}^{n \times n}$, where

   $$\lambda_1 = 2000\ , \quad \lambda_{n-1} = 2.1 \text{ and } \lambda_n = 2\ .$$

   Let σ be a shift value larger than λ_n. Give the shift and invert Power method to approximate λ_n. Give the rate of convergence for this method. (2.5 pt.)