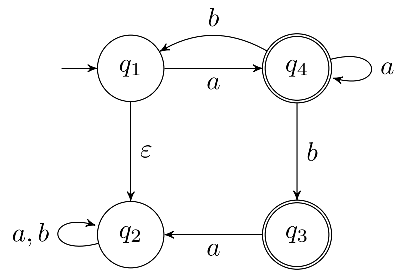
DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

Midterm Exam TI2316
Automata, Languages & Computability

May 31, 2017, 13:30–15:30

- Total number of pages (without this cover page): 4.
- This exam consists of 4 open questions of equal weight.
- Consulting handouts, readers, notes, books or other sources during this exam is prohibited. The use of electronic devices such as a calculator, mobile phones etc is also prohibited.
- A single exam cannot cover all topics, so do not draw conclusions based on this exam about topics that are never tested.
- Formulate your answers in correct English or Dutch and write legibly (use scrap paper first). Do not give irrelevant information, this could lead to a deduction of points.
- Before handing in your answers, ensure that your name and student number is on every page and indicate the number of pages handed in on (at least) the first page.

1. Suppose we have an NFA $N = (Q, \Sigma, \delta, q_1, F)$ with $\Sigma = \{a, b\}$ and the following transition graph:



(a) (1 point) Give $Q - F$.

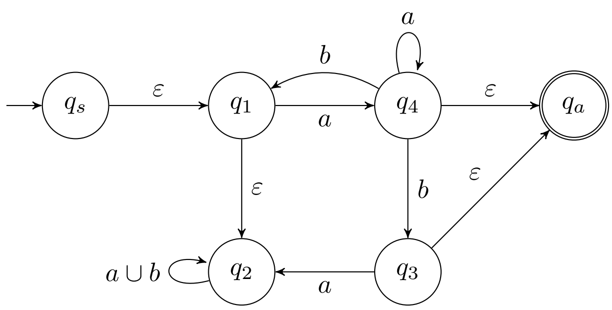
Solution: $\{q_1, q_2\}$

(b) (1 point) Give $\delta^*(q_1, abaab)$.

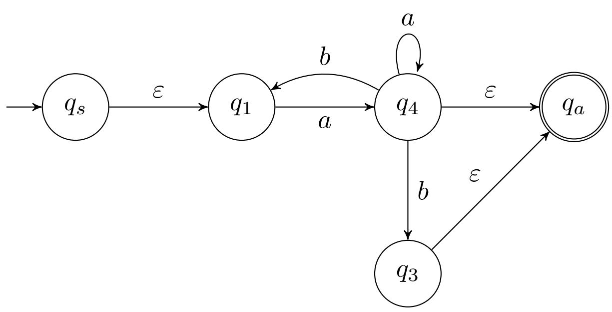
Solution: $\{q_1, q_2, q_3\}$

(c) (5 points) Use the method described by Sipser to find a regular expression R that is equivalent to N, i.e., such that $L(N) = L(R)$. Show every step in the process.

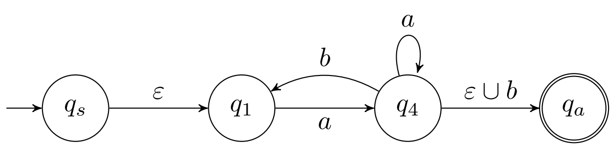
Solution: First, transform N into a GNFA:



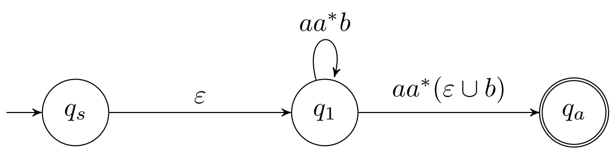
Then, eliminate states, first q_2:



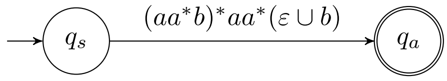
Then q_3:



Then q_4:



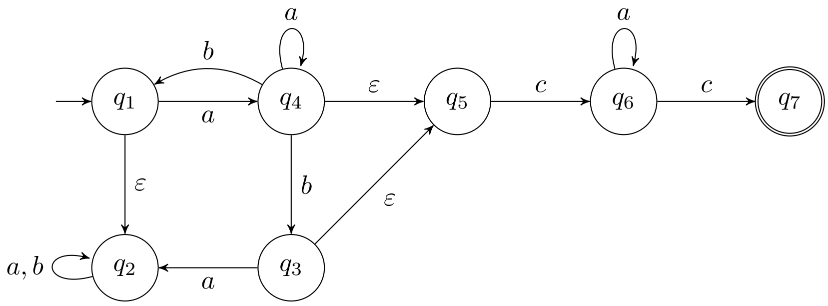
Finally, q_1:



Hence, $R = (aa^*b)^*aa^*(\varepsilon \cup b)$.

(d) (3 points) Give the transition graph of an NFA H such that $L(H) = L(N) \circ L(ca^*c)$ and H contains no more than 7 states.

Solution:



2. Suppose we have the following language over the alphabet $\Sigma = \{a, b, c\}$:

$$L = \{uv \mid u \in \Sigma^*, |u| = m \text{ and } v = a^n b^n c^n, \text{ with } m = 3n \text{ and } n \geq 0\}$$

(a) (7 points) Is L a regular language? If so, give a regular expression R such that $L(R) = L$. If not, prove this using the pumping lemma.

Solution: Suppose L is regular. This means there must exist some pumping length $p > 0$ for L such that all words w longer than p can be split up into three parts x, y and z, with $|y| > 0$ and $|xy| \leq p$. For this division of w, and any $i \geq 0$, $xy^iz \in L$. Let's take the word $w = a^{4p}b^pc^p$ which is in L ($m = 3p$ and $n = p$). This word is longer than p, so the above holds for this word. Given the requirements, we know that $x = a^\alpha, y = a^\beta$ and $z = a^{4p-\alpha-\beta}b^pc^p$, with $0 \leq \alpha < p$, $0 < \beta \leq p$ and $\alpha + \beta \leq p$. Now, taking $i = 2$, we get $a^\alpha a^{2\beta} a^{4p-\alpha-\beta} b^p c^p = a^{4p+\beta} b^p c^p$, which is clearly not in L since $\beta > 0$. We have obtained a contradiction, so L must not be regular.

(b) (3 points) Give a brief description (in no more than 15 lines) of how one might construct a DFA D that recognizes the difference of the languages of two NFAs N_1 and N_2, i.e., $L(D) = L(N_1) - L(N_2)$.

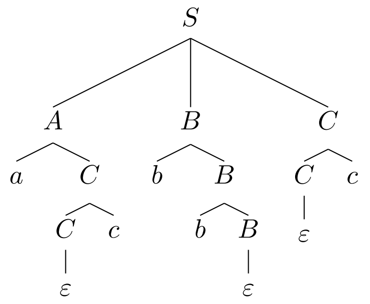
Solution: The difference can be defined in terms of intersection and complement: $L(D) = L(N_1) - L(N_2) = L(N_1) \cap L(N_2)^c$. First, we transform the NFAs into DFAs M_1 and M_2, respectively. We can then invert the accept states of M_2, resulting in M_2'. Then we use the construction for the intersection of M_1 and M_2' (proof of Theorem 1.25), choosing as accept states $F = F_{M_1} \times F_{M_2'}$.

3. Suppose we have a context-free grammar $G = (V, \Sigma, R, S)$, with R containing the following rules:

$S \to ABC$
$A \to aC \mid D$
$B \to bB \mid A \mid \varepsilon$
$C \to Ac \mid Cc \mid \varepsilon$
$D \to aa.$

(a) (1 point) Give a parse tree for the word $acbbc$.

Solution:



(b) (2 points) Give two different *leftmost* derivations of the word $aaaac$.

Solution:

$$S \Rightarrow ABC \Rightarrow DBC \Rightarrow aaBC \Rightarrow aaAC \Rightarrow aaaCC \Rightarrow aaaAcC \Rightarrow aaaaCcC \Rightarrow aaaacC \Rightarrow aaaac$$

$$S \Rightarrow ABC \Rightarrow DBC \Rightarrow aaBC \Rightarrow aaAC \Rightarrow aaDC \Rightarrow aaaaC \Rightarrow aaaaCc \Rightarrow aaaac$$

(c) (3 points) Give a grammar G' such that $L(G') = L(G)^*$.

Solution: $G' = (V, \Sigma, R', S)$, with R' containing the following rules:

$$S \to ABCS \mid \varepsilon$$
$$A \to aC \mid D$$
$$B \to bB \mid A \mid \varepsilon$$
$$C \to Ac \mid Cc \mid \varepsilon$$
$$D \to aa$$

(d) (4 points) Prove that if L_1 and L_2 are context-free languages, then L_1L_2 is also context free. (That is, context-free languages are closed under concatenation.)

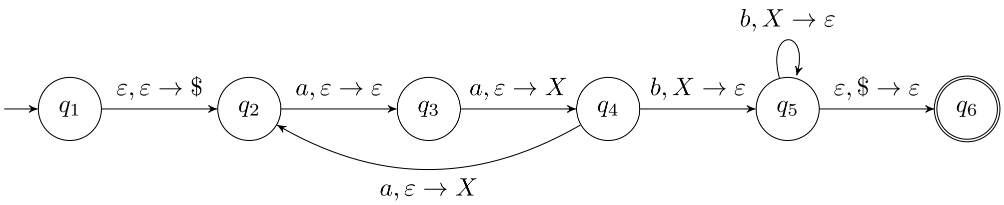
Solution: There exist context-free grammars $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$ for L_1 and L_2, respectively. A new grammar G that recognizes L_1L_2 can be created by combining G_1 and G_2 and adding a new start variable S, along with a new rule $S \to S_1S_2$. If necessary, rename variables from G_1 and G_2 to ensure $V_1 \cap V_2 = \emptyset$. Now $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \to S_1S_2\}, S)$.

4. Suppose we have the following language over the alphabet $\Sigma = \{a, b\}$:

$$L = \{a^m b^n \mid 2m = 3n + 1 \text{ and } m, n \geq 0\}$$

(a) (6 points) Give a PDA P that recognizes L, i.e., such that $L(P) = L$. A transition graph suffices. Your PDA must have no more than 9 states.

Solution:



(b) (2 points) Explain in no more than 10 lines how P recognizes L.

Solution: L can be written as $aa(aaa)^k(bb)^kb$, with $k \geq 0$. The PDA reads at least two as and additional as come in multiples of 3. Meanwhile, the stack keeps track of how many bs there will need to be: when there are 2 as, only 1 is needed, and otherwise 2 additional bs are needed for every 3 as. There is always at least 1 b due to the transition $q_4 \to q_5$. Finally, the PDA marks and checks for the bottom of the stack.

(c) (2 points) Show how P processes the word $aaaaabbb$ by giving a sequence of state descriptions.

Solution:

$$
\begin{aligned}
(q_1, aaaaabbb, \varepsilon) &\vdash (q_2, aaaaabbb, \$) \\
&\vdash (q_3, aaaabbb, \$) \\
&\vdash (q_4, aaabbb, X\$) \\
&\vdash (q_2, aabbb, XX\$) \\
&\vdash (q_3, abbb, XX\$) \\
&\vdash (q_4, bbb, XXX\$) \\
&\vdash (q_5, bb, XX\$) \\
&\vdash (q_5, b, X\$) \\
&\vdash (q_5, \varepsilon, \$) \\
&\vdash (q_6, \varepsilon, \varepsilon)
\end{aligned}
$$

End of the exam