Partial Exam 1 Advanced Algorithms

September 29, 2021, 9-12h

This partial exam consists of 6 questions on 10 pages. The total number of points is 60 . Your grade is determined by dividing the total score by 6. You are allowed to bring your own pens and a standard, non-graphical, calculator. No other devices such as smart phones, tablets, smartwatches, are allowed, nor are notes and books.

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1. (a) (8 points) Determine a basic feasible solution to the following problem by using Phase 1 of the Simplex algorithm. If there is a choice in entering basis variable, choose the variable with smallest index.

$$\begin{array}{llcl} \text{min } z = & x_2 & & \\ \text{s.t.} & -x_1 + x_2 & \leq & 2 \\ & -x_1 + x_2 & \geq & -2 \\ & x_1 + x_2 & \geq & 2 \\ & x_1, \quad x_2 & \geq & 0 \end{array}$$

Solution:

$$\begin{array}{ll} \text{min } z = & x_2 \\ \text{s.t.} & -x_1 + x_2 + s_1 = 2 \\ & x_1 - x_2 + s_2 = 2 \\ & x_1 + x_2 - s_3 + x_3^a = 2 \\ & x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ x_3^a \geq 0 \end{array}$$

Phase 1 objective function: $\min w = x_1^a = 2 - x_1 - x_2 + s_3$.

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	x_3^a
s_1	2	-1	1	1			
s_2	2	1	-1		1		
x_3^a	2	1	1			-1	1
$-w$	-2	-1	-1			1	

Choose x_1 as entering variable, and x_3^a as leaving basis variable. The Simplex tableau becomes:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	x_3^a	
s_1	4		2	1		-1	1	$r_1 + r_3$
s_2	0		-2		1	1	-1	$r_2 - r_3$
x_1	2	1	1			-1	1	r_3
$-w$	0						1	$r_0 + r_3$

Optimal solution to Phase 1 since $\bar{c}_j \geq 0$ for all j. Since $w = 0$ and the artificial variable is non-basic, the current bfs is feasible for the problem.

(b) (4 points) Is the basic feasible solution found in (a) optimal with respect to the original objective function? The answer "yes" or "no" accompanied by a short explanation suffices.

Solution: Reintroduce the original objective function $z = x_2$. The objective function is already expressed in non-basic variables. Since $\bar{c}_{x_2} > 0$ and all other $\bar{c}_j = 0$, the current bfs is optimal.

2. Given is the following optimization problem:

$$\begin{array}{lrcrcrcrcrcl}
\max z = & & & -4x_2 & + & 3x_3 & + & 2x_4 & - & 8x_5 & & \\
\text{s.t.} & 3x_1 & + & x_2 & + & 2x_3 & + & x_4 & & & = & 3 \\
 & x_1 & - & x_2 & & & + & x_4 & - & x_5 & \geq & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0,
\end{array}$$

(a) (6 points) Formulate the corresponding dual problem.

Solution:

$$\begin{array}{lrcrcr}
\min w = & 3\pi_1 & + & 2\pi_2 & & \\
\text{s.t.} & 3\pi_1 & + & \pi_2 & \geq & 0 \\
 & \pi_1 & - & \pi_2 & \geq & -4 \\
 & 2\pi_1 & & & \geq & 3 \\
 & \pi_1 & + & \pi_2 & \geq & 2 \\
 & & & -\pi_2 & \geq & -8 \\
 & \multicolumn{5}{l}{\pi_1 \in \mathbb{R},\ \pi_2 \leq 0}
\end{array}$$

(b) (3 points) Formulate all complementary slackness conditions.

Solution:

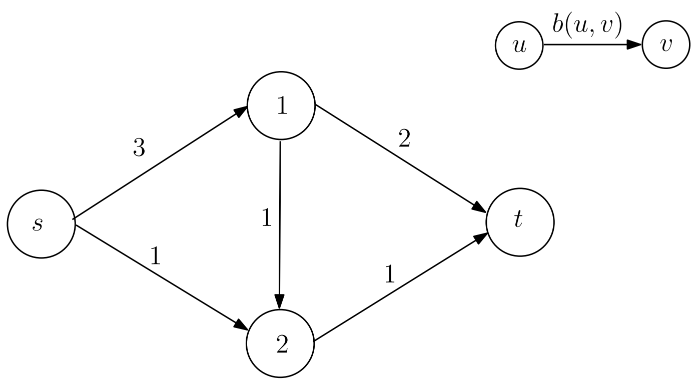
$$\begin{aligned}
(\pi_1(3-3x_1-x_2-2x_3-x_4) &= 0 && (1))\\
\pi_2(x_1-x_2+x_4-x_5-2) &= 0 && (2)\\
x_1(3\pi_1+\pi_2) &= 0 && (3)\\
x_2(\pi_1-\pi_2+4) &= 0 && (4)\\
x_3(2\pi_1-3) &= 0 && (5)\\
x_4(\pi_1+\pi_2-2) &= 0 && (6)\\
x_5(-\pi_2+8) &= 0 && (7)
\end{aligned}$$

Condition (1) does not provide any information and may be omitted.

(c) (3 points) The optimal solution to the dual problem is $(\pi^*)^T = (2, 0)$ objective value $w^* = 6$. Use the complementary slackness conditions from (b) to determine the optimal primal solution $x^*,\ z^*$.

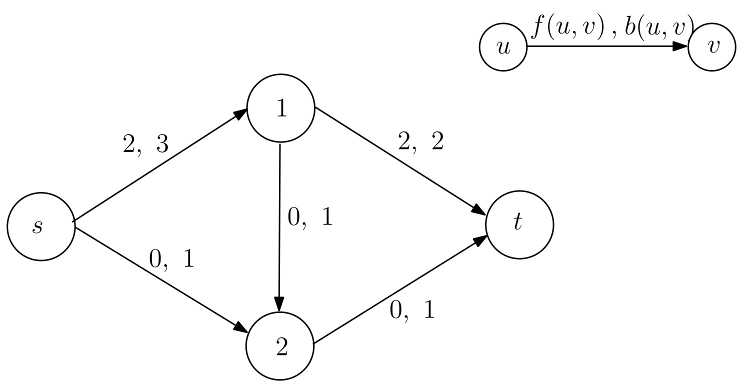
Solution: Since there is slack in all dual constraints except the fourth constraint, we get $x_1^* = x_2^* = x_3^* = x_5^* = 0$. Since $\pi_1^* \neq 0$, the first primal constraint is satisfied with equality (which it should be anyway in every feasible solution). Hence $3 - x_4^* = 0$, or $x_4^* = 3$. The optimal primal objective value is $z^* = 2x_4^* = 2 \cdot 3 = 6$.

3. (a) (2 points) Determine an upper bound on the maximum flow from s to t in the directed graph below. The capacities are given on the arcs. Specify how you have obtained the bound and motivate why it is an upper bound.

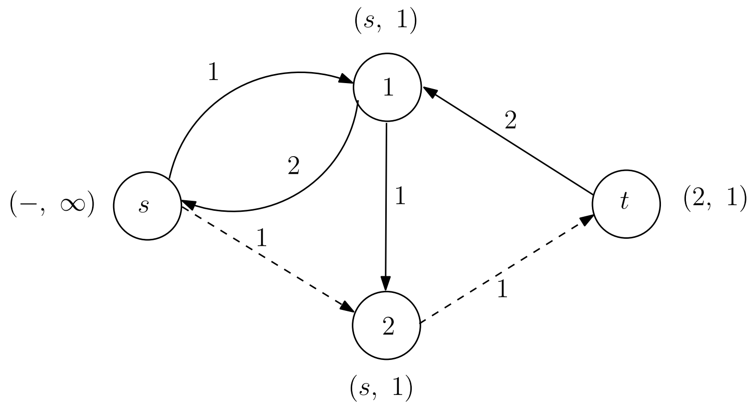


Solution: The capacity of any s-t cut in the graph provides an upper bound on the max flow from s to t. Take for instance the cut $W = \{s\}$, $\bar{W} = \{1, 2, t\}$. The capacity of the cut is $3 + 1 = 4$.

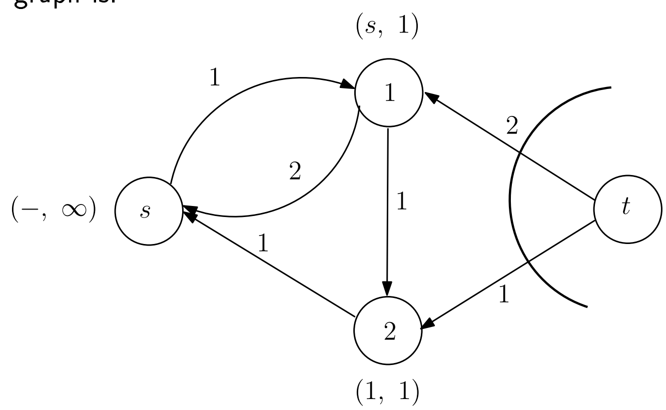
(b) (4 points) Consider the same directed graph as in (a). On the arcs you now see the current flow $f(a)$ and the capacities $b(a)$. Starting from this flow, determine the maximum flow in the graph using the Ford-Fulkerson algorithm. You can choose your augmenting path(s) as you wish. As an answer you can draw the graph and indicate the flow on each arc. Do not forget to mention the total flow.



Solution: In the auxiliary graph below, one augmenting path is: $s \to 2 \to t$ with capacity 1.



The new auxiliary graph is:



In this graph we see that there is no augmenting path any more, so the current flow is maximum. The flow on the arcs are as follows:

$$\begin{aligned} f(s,1) &= 2 \\ f(1,t) &= 2 \\ f(1,2) &= 0 \\ f(s,2) &= 1 \\ f(2,t) &= 1 \end{aligned}$$

The total flow from s to t is 3.

(c) (2 points) Give the minimum s-t cut and its capacity, which verifies the optimality of the flow you found in (b). You can either draw the cut in the graph, or you can give the partition of the vertex set $(W,\ \bar{W})$. Also, specify the cut capacity and how it is obtained.

Solution: The minimum s-t cut is indicated in the last graph is $W = \{s, 1, 2\},\ \bar{W} = \{t\}$. The vertices in W are the ones that can be reached from s in the final auxiliary graph. The cut capacity is the sum of the capacities on the arcs $(1,t)$ and $(2,t)$, i.e., $2 + 1 = 3$.

4. Consider the ILP below.

$$\begin{array}{llrcrcl} \max & z = & x_1 & + & 2x_2 & & \\ \text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\ & & -x_1 & + & x_2 & \leq & 3 \\ & & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\ & & \multicolumn{5}{l}{x_1, x_2 \in \mathbb{Z}} \end{array}$$

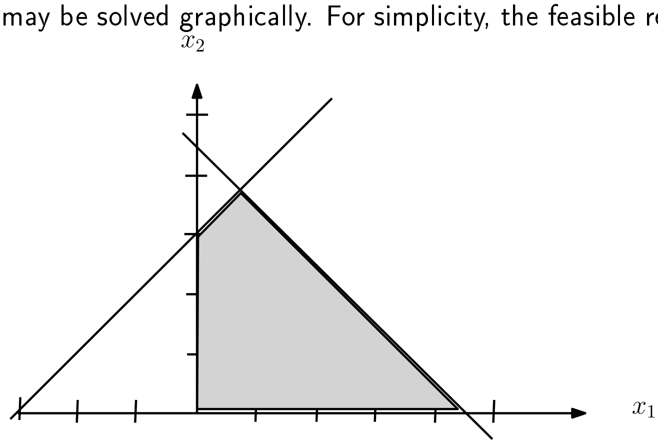
The optimal Simplex tableau of the LP relaxation is as follows:

basis	$\bar{b}$	x_1	x_2	s_1	s_2
x_1	$3/4$	1		$1/4$	$-1/2$
x_2	$15/4$		1	$1/4$	$1/2$
$-z$	$-33/4$			$-3/4$	$-1/2$

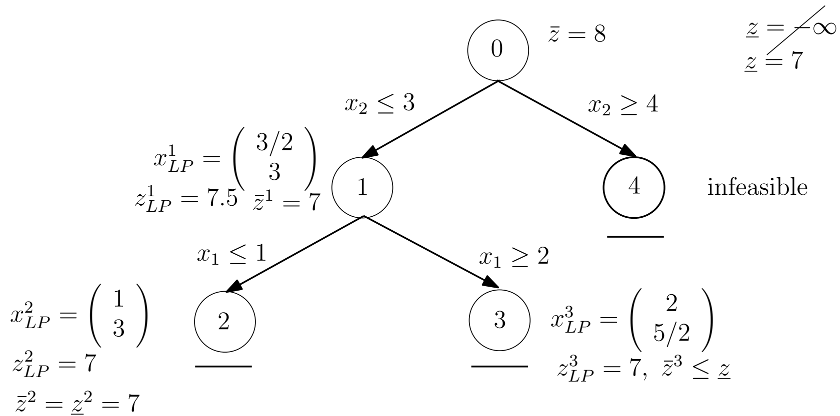
(a) (8 points) Determine an optimal solution to the ILP with the **branch & bound** method. Use the following search strategy:

- Whenever given a choice, branch first on variable x_2, then on x_1.
- Choose the $\leq$-branch first.
- Go depth first.

LP relaxations may be solved graphically. For simplicity, the feasible region is illustrated below.



Solution:
The branch and bound tree looks as follows:



The optimal solution is $(x^*)^T = (1, 3)$ with $z^* = 6$. Node 2 is pruned by optimality, node 3 by bound, and node 4 by infeasibility.

(b) (3 points) Derive a Gomory cut from the first row of the optimal Simplex tableau (the row in which variable x_1 is basic).

Solution: The first row reads:

$$x_1 + \frac{1}{4}s_1 - \frac{1}{2}s_2 = \frac{3}{4}\,.$$

Splitting the coefficients in integer and fractional parts yields:

$$x_1 + \frac{1}{4}s_1 + (-1 + \frac{1}{2})s_2 = \frac{3}{4}\,.$$

Integer parts left, and fractional parts right yields:

$$x_1 - s_2 = \frac{3}{4} - \frac{1}{4}s_1 - \frac{1}{2}s_2\,.$$

The Gomory cut is:

$$-\frac{1}{4}s_1 - \frac{1}{2}s_2 \leq -\frac{3}{4}\,.$$

5. Let $N := \{1, \dots, n\}$. Consider the integer knapsack problem

$$\begin{aligned} z = \max \sum_{j \in N} c_j x_j & & \\ \text{s.t.} \sum_{j \in N} w_j x_j &\leq b & \\ x_j &\geq 0, & \forall j \in N \\ x &\in \mathbb{Z}^n . & \end{aligned}$$

Assume that:

- $c_j, w_j,\ j \in N,\ b \in \mathbb{Z}_+$
- $w_j \leq b,\ j \in N$
- $\frac{c_1}{w_1} \geq \frac{c_j}{w_j},\ j \in N \setminus \{1\}$

The optimal solution to the LP-relaxation is:

$$x^{LP} = \begin{pmatrix} \frac{b}{w_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad z^{LP} = c_1 \cdot \frac{b}{w_1}$$

Let $\lfloor a \rfloor$ denote the value you obtain by rounding down a to the nearest integer. Consider the following simple algorithm:

Algorithm A
Input: The LP-solution as given above.
1. Set $x_j = \lfloor x_j^{LP} \rfloor$ for all $j \in N$.
Output: Integer vector x.

(a) (1 point) Argue that Algorithm A is a polynomial time algorithm.

Solution: Finding the element with the largest ratio c_j/w_j and then doing the rounding is polynomial. If one would “naively” solve the LP-relaxation, that would also be polynomial.

(b) (2 points) Show that the solution produced by Algorithm A is feasible for the integer knapsack problem.

Solution: Show that the solution x produced by A is feasible for the knapsack problem.

$$w_1 x_1 + \cdots + w_n x_n = w_1 \lfloor \frac{b}{w_1} \rfloor \leq w_1 \frac{b}{w_1} = b\,.$$

Moreover, $x \in \mathbb{Z}^n$, so x is feasible.

(c) (4 points) Show that the approximation guarantee of Algorithm A is 2, that is, show that Algorithm A is a 2-approximation algorithm.

Solution: Show that the approximation guarantee is 2. Let z^A be the value of the solution produced by Algorithm A. We have $x_1 = \lfloor b/w_1 \rfloor,\ x_2 = \cdots x_n = 0$, $z^A = c_1 \cdot \lfloor b/w_1 \rfloor$ and

$$\frac{z^A}{z^{LP}} = \frac{c_1 \cdot \lfloor b/w_1 \rfloor}{c_1 \cdot b/w_1} = \frac{\lfloor b/w_1 \rfloor}{b/w_1}\,.$$

Now, write $b/w_1 = \lfloor b/w_1 \rfloor + f$, where $0 \leq f < 1$. We now obtain

$$\frac{z^A}{z^{LP}} = \frac{\lfloor b/w_1 \rfloor}{\lfloor b/w_1 \rfloor + f}\,.$$

By assumption we have $\lfloor b/w_1 \rfloor \geq 1$. Combining with $0 \leq f < 1$ and $z^{LP} \geq z$ yields

$$\frac{z^A}{z} \geq \frac{z^A}{z^{LP}} = \frac{\lfloor b/w_1 \rfloor}{\lfloor b/w_1 \rfloor + f} > \frac{\lfloor b/w_1 \rfloor}{\lfloor b/w_1 \rfloor + \lfloor b/w_1 \rfloor} = \frac{1}{2}\,.$$

Since the problem is a maximization-problem, the approximation guarantee is

$$\frac{1}{\frac{1}{2}} = 2\,.$$

6. (10 points) A company has a set S of suppliers for a particular product. The company can supply the product to n different markets. The set of markets is denoted by M. For each supplier i we know to which subset $M_i \subseteq M$ of the markets he can deliver. The total supply of supplier i is s_i and the total demand of market j is d_j.

   If the company chooses to deliver to market j, it has to deliver the full amount d_j, and this demand may be delivered from multiple suppliers. The transportation costs per unit transported from supplier i to market j is c_{ij}. If the company chooses not to deliver to market j, the lost revenue is r_j.

   The company wants to find out how the suppliers should supply the products in such a way that the sum of the total transportation costs and the lost revenues are minimized. Formulate the problem as an mixed-integer linear optimization problem.

Solution: Let

$$x_{ij} = \text{ the amount of products transported from supplier } i \in S \text{ to market } j \in M\,,$$

$$z_j = \begin{cases} 1 & \text{if market } j \text{ is not served} \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
\min \sum_{i\in S}\sum_{j\in M_i} c_{ij}x_{ij} + \sum_{j\in M} r_j z_j \\
\text{s.t.} \sum_{\{i:j\in M_i\}} x_{ij} + d_j z_j &= d_j \quad \text{for all } j \in M \\
\sum_{j\in M_i} x_{ij} &\leq s_i \quad \text{for all } i \in S \\
z_j &\in \{0,1\} \quad \text{for all } j \in M \\
x_{ij} &\geq 0 \quad \text{for all } i \in S,\ j \in M_i
\end{aligned}
$$

End of exam