DELFT UNIVERSITY OF TECHNOLOGY
Faculty of EEMCS

Exam IN4344 Advanced Algorithms – Part 1 of 2

January 26, 2022, 13:30–16:30

- This is a closed-book individual examination with 6 questions worth 50 points in total.
- If your score is n points, then your grade for this exam part will be $n/5$.
- Use of a calculator is permitted.
- Write clearly, use correct English, and avoid verbose explanations. Giving irrelevant information may lead to a reduction in your score. *Almost all question parts can be answered in a few lines!*
- This exam covers the chapters and sections of the Syllabus mentioned on Brightspace in the content of Part 1.
- The total number of pages of this exam is 7 (excluding this front page).
- Exam prepared by K.I. Aardal.

1. (a) (4 points) Consider the following problem.

$$\begin{array}{rrcrcr} \max z = & x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 4 \\ & -x_1 & & & \geq & -1 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

i. Formulate the problem in standard form.

ii. Determine two basic feasible solutions corresponding to the extreme point $x^\mathsf{T} = (1, 2)$. For each solution, indicate which variables are basic.

Solution: i.

$$\begin{array}{rrcrcrcrcrcr} \max z = & x_1 & + & x_2 & & & & & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & s_1 & & & & & = & 4 \\ & x_1 & & & & & + & s_2 & & & = & 1 \\ & x_1 & + & x_2 & & & & & + & s_3 & = & 3 \\ & x_1, & & x_2, & & s_1, & & s_2, & & s_3 & \geq & 0 \end{array}$$

ii. We have 5 variables and 3 constraints. A basic solution is obtained by setting 5-3=2 variables to zero and solve for the remaining 3 variables. Here we know that $x_1 = 1,\ x_2 = 2$.
Set $s_1 = s_2 = 0$: The solution is $x_1 = 1,\ x_2 = 2,\ s_3 = 0$.
Set $s_1 = s_3 = 0$: The solution is $x_1 = 1,\ x_2 = 2,\ s_2 = 0$.

Two basic feasible solutions are (basic variables are bold):

$$x^1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ 0 \\ 0 \\ \mathbf{0} \end{pmatrix}, \quad x^2 = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ 0 \\ \mathbf{0} \\ 0 \end{pmatrix}.$$

(b) (5 points) Consider the problem

$$\begin{array}{ll} \min & -x_1 - 2x_2 - x_3 \\ \text{s.t.} & x_1 + 2x_2 \leq 2 \\ & x_1 + x_2 + x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

A helpful friend has started solving this problem by the simplex method and has reached the following tableau:

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_1	2	1	2	0	1	0
x_3	0	0	-1	1	-1	1
$-z$	2	0	-1	0	0	1

i. Indicate the pivot column and the pivot row, and do *one* simplex pivot. Give the resulting tableau.

ii. What can you conclude about the solution you obtained and why?

Solution: i.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_2	1	1/2	1	0	1/2	0
x_3	1	1/2	0	1	-1/2	1
$-z$	3	1/2	0	0	1/2	1

ii. This is a minimization problem. Since $\bar{c}_j \geq 0$ for all j, the solution is optimal.

2. Consider the following problem:

$$\begin{array}{lrcrcrcrcrcl}
\max z = & & - & 4x_2 & + & 3x_3 & + & 2x_4 & - & 8x_5 & & \\
\text{s.t.} & 3x_1 & + & x_2 & + & 2x_3 & + & x_4 & & & = & 3 \\
& x_1 & - & x_2 & & & + & x_4 & - & x_5 & \geq & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

(a) (5 points) Formulate the corresponding dual problem.

Solution:

$$\begin{array}{lrcrcr}
\min w = & 3y_1 & + & 2y_2 & & \\
\text{s.t.} & 3y_1 & + & y_2 & \geq & 0 \\
& y_1 & - & y_2 & \geq & -4 \\
& 2y_1 & & & \geq & 3 \\
& y_1 & + & y_2 & \geq & 2 \\
& & - & y_2 & \geq & -8 \\
& y_1 & & & & \text{free} \\
& & & y_2 & \leq & 0
\end{array}$$

(b) (3 points) Formulate all complementary slackness conditions corresponding to this specific primal-dual pair.

Solution:

$$\begin{aligned}
(y_1(3x_1 + x_2 + 2x_3 + x_4 - 3) &= 0) && (1)\\
y_2(x_1 - x_2 + x_4 - x_5 - 2) &= 0 && (2)\\
x_1(3y_1 + y_2) &= 0 && (3)\\
x_2(y_1 - y_2 + 4) &= 0 && (4)\\
x_3(2y_1 - 3) &= 0 && (5)\\
x_4(y_1 + y_2 - 2) &= 0 && (6)\\
x_5(-y_2 + 8) &= 0 && (7)
\end{aligned}$$

(c) (4 points) The optimal dual solution is

$$y^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

with objective value 6. Use the complementary slackness conditions that you derived in 2b) to determine the optimal primal solution and objective value.

Solution: Complementary slackness conditions (3)–(7) yields $x_1 = x_2 = x_3 = x_5 = 0$. Next, use the first primal constraint:

$$3x_1 + x_2 + 2x_3 + x_4 = 3.$$

Since only x_4 can be non-zero, we obtain $x_4 = 3$. Hence, the optimal primal solution is:

$$x^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

with $z^* = 6$.

3. (6 points) The problem of determining the shortest path from s to t in a directed graph $D = (V, A)$, where $s, t \in V$, can be formulated as a linear optimization problem:

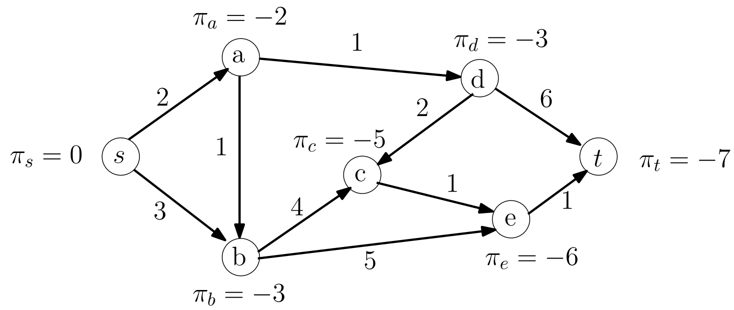
$$\begin{aligned} \min \quad & \sum_{a \in A} \ell_a f_a \\ \text{s.t.} \quad & Mf = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \\ & f \geq 0 \end{aligned}$$

Here, M is the node-arc incidence matrix with elements $m_{ij},\ i \in V,\ j \in A$, with $m_{ij} = 1$ if arc (i, j) leaves node i, and $m_{ij} = -1$ if arc (i, j) enters node j. The length of arc $a \in A$ is denoted by ℓ_a. The variable $f_a,\ a \in A$ denotes the "flow" on arc $a \in A$.

The corresponding dual problem is as follows.

$$\begin{aligned} \max \quad & \pi_s - \pi_t \\ \text{s.t.} \quad & \pi^T M \leq \ell^T \\ & \pi \in \mathbb{R}^{|V|} \end{aligned}$$

Consider the following directed graph $D = (V, A)$. We would like to determine the shortest path from s to t in D. The optimal *dual* solution $\pi(v)$, for all $v \in V$ is indicated in D.



i. *Use the dual solution* to determine the *length* of the shortest path from s to t. Give the length and shortly motivate how you determined it.

ii. *Use the dual solution and the complementary slackness conditions* to determine the shortest path from s to t. In your answer, give the path, and shortly motivate how you derived it.

Solution: i. The length of the shortest path is the optimal value of the primal problem. Since the problem is an LP, this value is equal to the dual optimal value. The dual optimal value is $\pi_s - \pi_t = 0 - (-7) = 7$. Hence the length of the shortest path is 7.

ii. Notice that the dual constraints are of the form

$$\pi_u - \pi_v \le \ell_{uv} \quad \forall (u, v) \in A\,.$$

Due to complementary slackness the following has to hold:

$$\begin{aligned} f_{uv} > 0 \quad &\Rightarrow \quad \pi_u - \pi_v = \ell_{uv} \\ \pi_u - \pi_v < \ell_{uv} \quad &\Rightarrow \quad f_{uv} = 0 \end{aligned}$$

So, we can take any path that uses arcs (u, v) that satisfy $\pi_u - \pi_v = \ell_{uv}$. The arcs (s, a), (a, d), (d, c), (c, e), and (e, t) satisfy this equation and hence form a shortest path.

4. Consider the following integer linear optimization problem

$$\begin{aligned} \max z = 4x_1 + 3x_2 & \\ \text{s.t. } 2x_1 + x_2 &\le 11 \\ -x_1 + 2x_2 &\le 6 \\ x_1,\ x_2 &\ge 0 \\ x_1,\ x_2 &\in \mathbb{Z} \end{aligned}$$

The optimal solution to the corresponding LP-relaxation is:

basis	$\bar{b}$	x_1	x_2	s_1	s_2
x_1	$\frac{16}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$
x_2	$\frac{23}{5}$	0	1	$\frac{1}{5}$	$\frac{2}{5}$
$-\bar{z}$	$-\frac{133}{5}$	0	0	$-\frac{11}{5}$	$-\frac{2}{5}$

(a) (2 points) Determine a Gomory cut from the row in which x_2 is basic.

Solution: The row reads: $x_2 + (1/5)s_1 + (2/5)s_2 = 23/5$. All the integer parts in the left-hand side, and the fractional parts in the right-hand side:

$$x_2 - 5 = 3/5 - (1/5)s_1 - (2/5)s_2.$$

Since the left-hand side is integer in any feasible solution, the right-hand side should be integer as well and since $s_1, s_2 \geq 0$, we obtain the Gomory cut

$$3/5 - (1/5)s_1 - (2/5)s_2 \leq 0$$

or

$$-(1/5)s_1 - (2/5)s_2 \leq -3/5.$$

(b) (4 points) Add the Gomory cut to the current tableau after introducing a slack variable, and re-optimize using the dual simplex method. What is the new solution and objective value?

Solution: Add a slack variable to the Gomory cut: $-(1/5)s_1 - (2/5)s_2 + s_3 = -3/5$. Introduce this row to the tableau:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_1	$\frac{16}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	0
x_2	$\frac{23}{5}$	0	1	$\frac{1}{5}$	$\frac{2}{5}$	0
s_3	$-\frac{3}{5}$	0	0	$-\frac{1}{5}$	$-\frac{2}{5}$	1
$-z_{LP}$	$-\frac{133}{5}$	0	0	$-\frac{11}{5}$	$-\frac{2}{5}$	0

s_3 is the leaving variable and s_2 the entering variable. After one iteration of dual simplex we obtain:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_1	3.5	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
x_2	4	0	1	0	0	1
s_2	1.5	0	0	$\frac{1}{2}$	1	$-\frac{5}{2}$
$-z_{LP}$	$-\frac{130}{5}$	0	0	-2	0	-1

The new solution is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix}$$

with objective value $z_{LP} = 130/5 = 26$.

5. (10 points) As a leader of an energy exploration company you must determine the least-cost *selection of 5 out of 10 possible projects*. Label the projects $P_1, \ldots, P_{10}$. The exploration costs associated

with the projects are $C_1, \ldots, C_{10}$ respectively.

Regional development restrictions are such that:

1. Exploring projects P_1 and P_7 will prevent you from exploring P_8
2. Exploring projects P_3 or P_4 prevents you from exploring P_5.
3. Of the group of projects P_5, P_6, P_7, P_8, at most two projects may be explored.

Formulate an *integer linear* optimization problem to determine the minimum-cost exploration scheme of five projects that satisfies these restrictions.

In your answer, give a clear variable definition, and make a short comment explaining each of the constraints.

Solution: Let

$$x_j = \begin{cases} 1 & \text{if project } j \text{ is chosen} \\ 0 & \text{otherwise.} \end{cases} \quad , \quad y = \begin{cases} 1 & \text{if projects } P_1 \text{ and } P_7 \text{ are both chosen} \\ 0 & \text{otherwise.} \end{cases}$$

$$\min \sum_{j=1}^{10} C_j x_j + y \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{10} x_j = 5 \tag{2}$$

$$x_1 + x_7 - y \leq 1 \tag{3}$$

$$x_8 - (1 - y) \leq 0 \tag{4}$$

$$x_5 - (1 - x_3) \leq 0 \tag{5}$$

$$x_5 - (1 - x_4) \leq 0 \tag{6}$$

$$x_5 + x_6 + x_7 + x_8 \leq 2 \tag{7}$$

$$x_j \in \{0, 1\}, \ j = 1, \ldots, 10 \tag{8}$$

$$y \in \{0, 1\}. \tag{9}$$

Expression (1) is the objective function minimizing the costs. Expression (2) forces us to select precisely five of the projects. Expressions (3)–(4) models restriction 1. Notice, we have added y in the objective function in order to avoid the situation $x_1 + x_7 \leq 1$ combined with $y = 1$. Since y has a positive cost, it will only be put to 1 if $x_1 + x_2 = 2$. Expression (5)–(6) models restriction 2. Expression (7) models restriction 3. Expressions (8)–(9) restrict the variables to take values 0 or 1 only.

6. We are given an integer linear minimization problem P.

   (a) (4 points) Suppose we have a specific instance of P, and that for this instance the optimal solution to the integer problem is equal to 3 and the optimal solution to the linear relaxation is equal to $3/2$. Given this information, give a bound on the performance guarantee of any approximation algorithm that is based on rounding a solution to the linear relaxation. Do not forget to motivate your answer.

Solution: For ρ to be a performance guarantee of an approximation algorithm, then for any instance the following has to hold:

$$z_{IP} \leq z_{ALG} \leq \rho \cdot z_{LP} .$$

So, $\rho \geq z_{IP}/z_{LP}$ for all instances. For our example we have $z_{IP}/z_{LP} = 3/(3/2) = 2$, which implies that $\rho \geq 2$, so the performance guarantee of an algorithm based on the LP-relaxation is at least equal to 2.

(b) (3 points) Suppose we have an algorithm for P that always produces, in polynomial time, a feasible solution with value z_{ALG}, such that $z_{ALG} \leq \rho \cdot z_D$, where z_D is the optimal value of the dual of the LP-relaxation of P. Argue why it holds that the algorithm is a ρ-approximation algorithm for P.

Solution: $z_{ALG} \leq \rho \cdot z_D = \rho \cdot z_{LP} \leq \rho \cdot z_{IP}$, so $z_{ALG} \leq \rho \cdot z_{IP}$. Here z_{LP} is the optimal value of the LP-relaxation, and z_{IP} is the optimal value of P. The equality holds due to the strong duality theorem, and the last inequality holds since the LP-relaxation is a relaxation of P. Since the algorithm runs in polynomial time and $z_{ALG} \leq \rho \cdot z_{IP}$, the algorithm is a ρ-approximation algorithm for P.

End of test