Endterm Reasoning and Logic (CSE1300)

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Please read the following information carefully!

- This exam consists of 20 multiple-choice questions and 6 open questions.
- The points for the multiple-choice part of the exam are computed as $1 + 9 \cdot \max(0, \frac{\text{score} - 0.25 * 20}{0.75 * 20})$. This accounts for a 25% guessing correction, corresponding to the four-choice questions we use.
- The grade for the open questions is computed as: $1 + 9 \cdot \frac{\text{score}}{66}$.
- The **final grade for the exam** is computed as: $0.4 \cdot MC + 0.6 \cdot Open$.
- This exam corresponds to all chapters of the book: *Delftse Foundations of Computation* (version 1.01).
- You have 3 hours to complete this exam.
- Before you hand in your answers, check that the sheet contains your name and student number, both in the human and computer-readable formats.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Note that the order of the letters next to the boxes on your multiple-choice sheet may **not always be A-B-C-D**!
- Tip: mark your answers on this exam **first**, and only after you are certain of your answers, copy them to the multiple-choice answer form.
- Read every question properly and in the case of the open questions, give **all information** requested, which should always include a brief explanation of your answer. Do not however give irrelevant information – this could lead to a deduction of points.
- Note that the minimum score per (sub)question is 0 points.
- You may write on this exam paper and take it home.
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Open questions:

Question:	21	22	23	24	25	26	Total:
Points:	11	8	11	11	13	12	66

Learning goals coverage, based on the objectives of all lectures (strongly paraphrased):

Goal	mt 17	et 17	mc 18	mt 18	et 18
translate logic to and from natural language			1,2	3,4	1
describe $\wedge$,$\vee$,$\neg$,$\rightarrow$, and $\leftrightarrow$ operators	1				
construct a truth table	2, 11a		3-5	1a,1b	31a
determine prop. logic equivalence	11b		6,7,19		2
rewrite logical connectives	6		8-10		31b
describe contrapositives, converses, and inverses.	3		11,12		
describe logic validity			13,14		3
describe sufficient and necessary conditions	4		15		4
prove validty of argument in prop. logic	5		16-17	1b	
describe the principle of explosion			18	1c	
explain why prop. logic is not sufficiently expressive			20		
describe $\forall$ and $\exists$ quantifiers	7,8		21	2c	5
evaluate negation stmt. in pred. logic			22		
construct a Tarski's world			23-25		
construct a formal structure in pred. logic	12		26-27	2b	32a
evaluate claims about formal structures	12		28-29	2a	6,32b
construct counterexamples for claims	10		30	2a,2b,5c	
describe the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$					
describe the form of a proof by division into cases				5b	7
describe the form of a proof by contradiction					
construct a proof by division into cases				7a	
construct a proof by contradiction	9				
explain what a theorem prover is.					8
describe the form of a proof by contrapositive				5a	
construct a proof by contrapositive	13			7b	
describe the form of a proof by generalisation.				5a,5b	
construct a proof by generalisation					
construct an existence proof					9
identify type of proof to use for a given claim				5b	
compute a sequence given a recursive definition				6a	10
construct and interpret recursive definitions		3		6b,6c	
explain the basic principle of an induction proof		2			11
construct an induction proof for a claim about numbers					33a
construct an induction proof for algorithms		4			33b
construct recursive definitions on sets		12a			12,13
construct a proof using structural induction		12b			14,15
explain and apply basic set operations.		1			16
construct Venn diagrams		5			17,18
construct formal counterexamples for claims on sets		1,13			19,34b
compute the powerset of a set					20,21
compute the cartesian product of two sets					22
construct proofs for claims on sets					34a
describe Cantor's proofs about infinite sets		11b			23
construct a function or relation from nat. language					24,25
describe the diff. between a function and a relation					35a
determine the inverse of relations and functions		8			35b
determine if a function is well-defined		6			26
determine if a function is injective, surjective, or bijective		7,11a,11c			27
determine if a relation is symmetric, transitive or reflexive		9			28,29
describe the properties of an equivalence relation		10			30

Multiple-Choice questions

1. (1 point) Which of the following claims **cannot** be **true** for two sets A and B?
   A. $\mathscr{P}(A) \subseteq A$
   B. $A \subseteq \mathscr{P}(A)$
   C. $A \cup \{1\} \in \mathscr{P}(A \cup \{1\})$
   D. $A \cup B \in \mathscr{P}(A) \cup \mathscr{P}(B)$

   Answer: Take $A = B = \emptyset$, now both B and D are already true. C is true for every A. The first claim cannot be true. Consider the following informal argument: $|\mathscr{P}(A)| = 2^{|A|} > |A|$ for all $|A| \geq 0$.

2. (1 point) Which of the following translations of the statement 'Sean and Michelle solve several different crimes.' to predicate logic is **most** accurate? The predicates used are:

   - s for Sean.
   - m for Michelle.
   - $Crime(x)$ for x is a crime.
   - $Solve(x, y)$ for x solves y.

   A. $\exists x(Crime(x) \wedge (Solve(s, x) \vee Solve(m, x)))$
   B. $\forall x(Crime(x) \rightarrow (Solve(s, x) \rightarrow Solve(m, x)))$
   C. $\exists x, y(Crime(x) \wedge Crime(y) \wedge (x \neq y) \wedge (Solve(s, x) \wedge Solve(m, x)) \wedge (Solve(s, y) \wedge Solve(m, y)))$
   D. $\forall x(Crime(x) \wedge \exists y(Crime(y) \wedge (x \neq y) \wedge (Solve(s, x) \wedge Solve(m, x)) \wedge (Solve(s, y) \wedge Solve(m, y))))$

   Answer:
   A. This only states they solved one crime.
   B. This allows for them to have solved no crimes at all.
   C. This states they solved at least two different crimes. Two is not necessarily many, but at least they are two different crimes and are guaranteed to exist.
   D. This uses the $\forall$ with $\wedge$ meaning that all objects are crimes.

   So C is the closest answer.

3. (1 point) Consider the following argument:

   $A \subseteq B$
   $(B \cap C) \neq \emptyset$

   $\therefore A \subseteq (B \cap C)$

   Which of the following presents a counterexample to this argument?
   A. $A = B = C = \{1, 2, 3\}$
   B. $A = B = \{1, 2, 3\}, C = \{1, 2\}$
   C. $A = C = \{1, 2, 3\}, B = \{1, 2\}$
   D. $B = C = \{1, 2, 3\}, A = \{1, 2\}$

   Answer:
   A. The conclusion is true here.
   B. Both premises hold, but the conclusion is false.
   C. The first premise is false here.
   D. The conclusion is true here.

4. (1 point) Which of the following functions is **not** well-defined?

   A. $L : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ **with** $L(n, m) = \frac{n+4}{m-3}$**.**

   B. $I : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ with $I(n, m) = n^{m+3}$.

   C. $N : \mathbb{N} \to \mathbb{Z}$, with $N(n) = \begin{cases} -n & \text{if } n \text{ is odd} \\ n+1 & \text{else} \end{cases}$.

   D. $K : \mathbb{R} \to \mathbb{Q}$ with $K(n) = \frac{\lceil n \rceil}{2}$. Note that $\lceil x \rceil$ is to round x up to the closest integer.

Answer: Take $m = 3$.

5. (1 point) Consider the following templates for a proof by induction. Which one does **not** prove their claim over natural numbers? Note: The claims also differ from answer to answer.

   A. Claim: $\forall n \geq 0(P(n))$.
   Base case: prove $P(0) \wedge P(1)$.
   Inductive step: prove $P(k) \to P(k+2)$ holds for all $k \geq 0$.

   B. Claim: $\forall n \geq 5(P(n))$**.**
   Base case: prove $P(0) \wedge P(3)$**.**
   Inductive step: prove $P(k) \to P(k+1)$ **holds for all** $k \geq 5$**.**

   C. Claim: $\forall n \geq 5(P(n))$.
   Base case: prove $P(5)$.
   Inductive step: prove $\neg P(k+1) \to \neg P(k)$ holds for all $k \geq 5$.

   D. Claim: $\forall n \geq 0(P(n))$.
   Base case: prove $P(0)$.
   Inductive step: prove $P(k+1) \to P(k+2)$ holds for all $k \geq -1$.

Answer:

   A. This works, as the base cases cover the odd and even numbers respectively.

   B. This does not work. The induction step only starts at $P(5)$ **which we have never proven to hold.**

   C. This works, as the second half is just the contrapositive of what we should prove.

   D. This works, the $+1$ on both sides is irrelevant. With $k = -1$ we get: $P(0) \to P(1)$ which gets our chain started.

6. (1 point) Which of the following **cannot** be used to show that two propositions P and Q are equivalent?

   A. Show that $(P \to Q) \wedge (Q \to P)$ is a tautology.

   B. Show that $\neg Q \to \neg P$ **and** $\neg P$ **are both contradictions.**

   C. Show that $(P \to R) \wedge (R \to Q) \wedge (Q \to P)$ is a tautology.

   D. Show that P is a contradiction and that $\neg Q$ is a tautology.

Answer:

   A. This shows that $P \leftrightarrow Q$ is a tautology, which is a correct method to show that $P \equiv Q$.

   B. This is nonsense, it shows that P **and** $\neg Q \wedge P$ **are tautologies. This in fact proves that** Q **is a contradiction and that** P **is a tautology, so they are not equivalent.**

   C. This proves that $P \leftrightarrow Q$ ($P \to Q$ due to transitivity, and $Q \to P$ is already listed) is a tautology, which is a correct method to show that $P \equiv Q$.

D. This implies that Q is also a contradiction and as all contradictions are equivalent, this shows that $P \equiv Q$.

7. (1 point) Consider the following informal description of a relation R. R is a binary relation from the set $A = \{0, 1, 2\}$ to the set $B = \{a, b, c\}$, in such a way that every element from A has a relation to exactly 2 unique items in B.

Which of the following describes R in set-notation?

A. $R = A \times B$

B. $R = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}$

C. $R = \{(0, 1), (1, 2), (2, 0), (a, a), (b, b), (c, c)\}$

D. $R = \{(a, 0), (b, 0), (a, 2), (b, 1), (c, 2), (c, 1)\}$

Answer: Every element in A needs to be mapped to 2 unique elements in B. This is exactly what answer B does. Answer A would map every element to 3 other elements. Answer C is not from set A to set B and answer D maps A to three unique items (namely to all of B).

8. (1 point) Consider the argument: $\dfrac{\begin{matrix}A\\B\end{matrix}}{\therefore C}$. Which of the following methods can we always use to prove the validity of the argument?

A. Show that $\dfrac{A}{\therefore C}$ is a valid argument.

B. Show that $\neg(A \to B) \to \neg C$ is a tautology.

C. Use equivalence rules to derive that $A \equiv B$ and that B is a tautology.

D. Construct a truth table for A, B, and C and inspect all rows in which C is false. The argument is valid if and only if both A and B are also false in those rows.

Answer:

A. This just means that B is a premise that we do not need, but the argument is still valid.

B. This does not exclude the situation $A = B = 1$ $C = 0$, which is the only counterexample to the argument and should thus be excluded to conclude validity.

C. This only holds if C is also a tautology, which we do not know.

D. Either A or B being false is sufficient.

9. (1 point) Which of the following claims about sets A, B, C is always **true**?

A. $|(A \cap B) \cup C| \geq |A \cap (B \cup C)|$

B. $|A \cup B| > |A|$ and/or $|A \cup B| > |B|$

C. $|A \cap B| \geq |A \cap C|$ if ($|B| \geq |C|$ and $|A| > 0$)

D. $|A \cap B| > 0$ if and only if ($|A| > 0$ and $|B| > 0$)

Answer:

A. Correct as $(A \cap (B \cup C)) \subseteq ((A \cap B) \cup C)$**.**

B. Take $A = B = \emptyset$.

C. Take $A = C = \{1\}$ and $B = \{2\}$.

D. Take $A = \{1\}$ and $B = \{2\}$.

10. (1 point) In his TED talk "What do top students do differently?"[1] Douglas Barton claims that: "In our research we found that hard work was a necessary condition, but it wasn't a sufficient condition to doing well."

Taking p to mean 'working hard' and q to mean 'doing well', which of the following is true?

A. $p \leftrightarrow q$

B. $p \rightarrow q \land \neg(q \rightarrow p)$

C. $\neg(p \rightarrow q) \land (\neg p \rightarrow \neg q)$

D. The claim is a contradiction, and can thus be expressed as $\mathbb{F}$.

Answer: Barton claims that $\neg(p \rightarrow q)$ holds (it is not sufficient) and that $q \rightarrow p$ holds (it it necessary). Answer C expresses this, using the contrapositive of $q \rightarrow p$.

11. (1 point) We want to prove the following claim over all natural numbers $\mathbb{N}$: $\forall n(4 \mid f(n) \lor 3 \mid f(n))$, where $f(n)$ is some function $f : \mathbb{N} \rightarrow \mathbb{N}$. What proof-outline sketched below is **not** suitable?

A. Take an arbitrary k, show that $12 \mid f(k)$.

B. Take an arbitrary k, show that $3 \nmid f(k) \rightarrow 4 \mid f(k)$ holds.

C. Take an arbitrary k**, show that** $2 \mid k \rightarrow 4 \mid f(k)$ **holds and that** $3 \mid k \rightarrow 3 \mid f(k)$ **holds.**

D. Take an arbitrary k, show that when $k = 2m$ it holds that $3 \mid f(k)$, and that when $k = 2m+1$ it holds that $4 \mid f(k)$.

Answer:

A. $12 \mid f(k) \rightarrow (3 \mid f(k) \land 4 \mid f(k))$ holds.

B. This is a rewriting of the $\lor$ statement.

C. This is not a valid proof as it does not cover all cases.

D. This shows that for al numbers one or the other holds, which is what we want to show.

12. (1 point) Consider the following formal structure S with domain $\{p, a, m, f, k, g\}$:

- $L^S = \{p, a, g\}$
- $P^S = \{m, f, k\}$
- $W^S = \{(p, m), (p, f), (m, p), (a, p), (a, k), (k, p)\}$
- $B^S = \{(p, p), (m, m), (a, p), (g, g), (k, m)\}$

Which of the following claims is **true**?

A. $\exists x(L(x) \land \forall y(W(x, y)))$

B. $\forall x(L(x) \rightarrow \exists y(B(x, y) \land (x \neq y))$

C. $\forall x((L(x) \rightarrow P(x)) \lor (P(x) \rightarrow L(x)))$

D. $\exists x(P(x) \land \forall y((P(x) \land (x \neq y)) \rightarrow B(x, y)))$

[1] https://www.youtube.com/watch?v=Na8m4GPqA30

Answer:

A. For $x = y$ is a counterexample for all choices for x.

B. Take $x = y = g$ as a counterexample.

C. This is true. As $P(x)$ is false for $x \in \{p, a, g\}$ and $L(x)$ is false for $x \in \{m, f, k\}$. This means one of the implication always holds.

D. For $x = m$ take $y = a$, for $x = f$ take $y = g$ and for $x = k$ take $y = a$ for a counterexample among many.

13. (1 point) Consider the following recursive definition of a set of strings ('words') S:

I. $a \in S$.

II. if $x \in S$, then $xi \in S$, $axa \in S$.

III. if $x, y \in S$, then $ixiyixi \in S$.

IV. Nothing else is in S other than the strings constructed with these rules.

Note that in this definition a and i are letters of the word and that x and y are variables, they represent arbitrary words in the set.

Which of the following claims over this set is **false**?

A. All words in the set have at least one a in it.

B. There is a word in the set without an i in it.

C. There is a word in the set with exactly five a's in it.

D. All words in the set have an even numbers of i's in it.

Answer:

A. Since a is the base case, this is true.

B. For instance the base case word.

C. Take $iaaaiaiai$ which you can create by applying rule II to a first, then apply rule III to aaa and a.

D. This is false. We can for instance create ai by applying rule II once.

14. (1 point) Consider again the recursive definition from the previous Question 13. Which of the following rule(s) should we add to ensure that string mia is a part of the language?

A. V. if $x \in S$, then $mx \in S$.

B. V. $i \in S$. VI. if $x \in S$, then $xm \in S$

C. V. $m \in S$. VI. if $x, y \in S$, then $xiy \in S$.

D. V. $m \in S$. VI. if $x, y \in S$, then $ixy \in S$.

Answer: Only the answer allows us to create mia. With both m and a in the language, we can just apply xiy with $x = m$ and $y = a$.

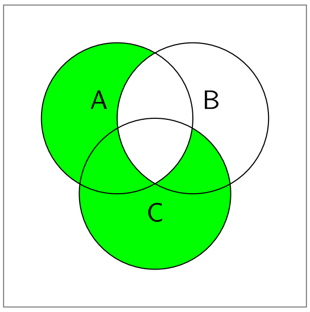


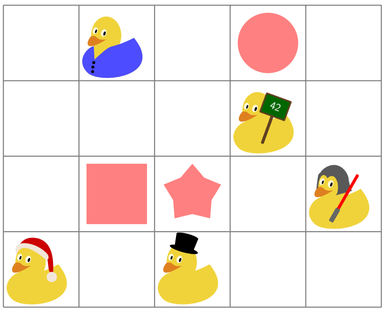
Figure 1: Venn Diagram

15. (1 point) Figure 1 shows a Venn Diagram. Which of the following sets is represented by the shaded area?

A. $(A\Delta B) \cup (C\Delta B)$
B. $(A \setminus B) \cup (C \setminus A)$
C. $(A^c \setminus B) \cup (C \setminus (A \cap B))$
D. $((C \setminus A) \cap (C \setminus B)) \cup (A\Delta B)$

Answer:
A. This would require $B - A$ to be fully shaded.
B. This is correct.
C. This would require the area outside of A,B,C to be shaded.
D. Again this would require $B - A$ to be shaded.

16. (1 point) Consider now the following alternative visualisation of sets: a Duck Empire. The set P contains all ducks that are within a distance of 2 from the circle, Q contains all ducks within a distance of 2 from the star, and R contains all ducks within a distance of 2 from the square.[2]



Which of the following claims is **true**?
A. $|P \cap Q| = 5$
B. $|Q \cup R| = 5$
C. $|R \times P| = 5$
D. $|P| + |Q| + |R| = 5$

[2]Ducks cannot travel diagonally, so the duck with the Santa hat is exactly 2 squares always from the square, but a distance 3 away from the star.

Answer: $P = \{\text{sign,jacket}\}, Q = \{\text{sign,hat,sabre}\}$, and $R = \{\text{hat,jacket,santa}\}$.

17. (1 point) Which of the following statements is **true**?
    - A. $\mathscr{P}(\emptyset) = \emptyset$
    - **B.** $\mathscr{P}(\emptyset) = \emptyset \cup \{\emptyset\}$
    - C. $\mathscr{P}(\mathscr{P}(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
    - D. $\mathscr{P}(\mathscr{P}(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\}$

Answer: Based on size alone we can see that:
- A. $|\mathscr{P}(\emptyset)| = 2^0 = 1$ So incorrect.
- **B.** $|\mathscr{P}(\emptyset)| = 2^0 = 1$**, It is also this set that is given here.**
- C. $|\mathscr{P}(\mathscr{P}(\emptyset))| = 2^{2^0} = 2$, but this set is of size 4.
- D. $|\mathscr{P}(\mathscr{P}(\emptyset))| = 2^{2^0} = 2$, but this set is of size 3.

18. (1 point) Which of the following claims is **false**?
    - A. $\mathbb{N}$ has the same cardinality as $\{\pi^k \mid k \in \mathbb{Q}\}$.
    - B. $\mathbb{N}$ has the same cardinality as $\{x \mid x = 42k$ for some $k \in \mathbb{Z}\}$.
    - **C. $\mathbb{N}$ has the same cardinality as $\{x \mid 0.1 \leq x \leq 0.11$ for $x \in \mathbb{R}\}$.**
    - D. $\mathbb{N}$ has the same cardinality as $\{x \mid x = 2k + 1$ for some $k \in \mathbb{N}\}$.

Answer:
- A. This is true as $\mathbb{Q}$ has the same cardinality as $\mathbb{N}$. Both are countably infinite.
- B. This is true as $\mathbb{Z}$ has the same cardinality as $\mathbb{N}$. Both are countably infinite.
- **C. This is true as $\mathbb{R}$ has a larger cardinality than $\mathbb{N}$, and this is also true for a small range of numbers in $\mathbb{R}$ like this.**
- D. This is true. As discussed in the lectures there are just as many even numbers (but also odd nubmers) as natural numbers. Both are countably infinite.

19. (1 point) Which of the following claims is **true** ?
    - A. A relation that is not reflexive, cannot be transitive.
    - B. A relation that is not transitive, cannot be symmetric.
    - C. A relation that is both symmetric and transitive, has to be reflexive.
    - **D. All 8 permutations of relations that are (not) symmetric, (not) transitive and/or (not) reflexive can be created.**

Answer: We can indeed create any such relation. Take:

- Reflexive, symmetric, transitive $D = \{a\}, R = \{(a, a)\}$
- Reflexive, symmetric, not transitive $D = \{a, b, c\}, R = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$
- Reflexive, not symmetric, transitive $D = \{a, b, c\}, R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$
- Reflexive, not symmetric, not transitive $D = \{a, b, c\}, R = \{(a, a), (b, b), (c, c), (a, b), (b, c)\}$
- Not reflexive, symmetric, transitive $D = \{a\}, R = \{\}$
- Not reflexive, symmetric, not transitive $D = \{a, b, c\}, R = \{(a, b), (b, c)\}$
- Not reflexive, not symmetric, transitive $D = \{a, b\}, R = \{(a, b)\}$
- Not reflexive, not symmetric, not transitive $D = \{a, b, c\}, R = \{(a, b), (b, c)\}$

20. (1 point) As Lyra prepares to travel to the north, she draws up a map of the kingdom of the Panserbjørn (see Figure 2). On it she also draws the different roads available between the cities of the kingdom. Lyra's friend Pantalaimon realises this could also be a visualisation of a relation $P(x, y)$. Which of the following statements about this relation is true?

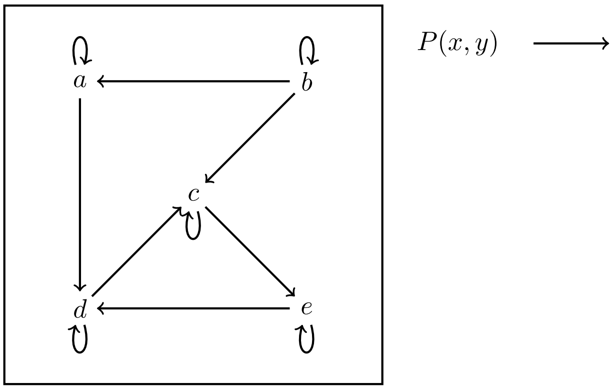


Figure 2: The map as drawn by Lyra

A. P is not currently transitive, but adding $P(a, c)$ and $P(b, e)$ would make it transitive.

B. P is currently reflexive, but removing $P(b, c)$ and adding $P(c, b)$ would make it lose its reflexivity.

C. P is currently transitive, but removing $P(a, d)$ and $P(e, d)$ and adding $P(a, c)$ and $P(c, d)$ from/to the relation would make it lose this property.

D. P is not currently symmetric, but removing $P(b, c), P(b, a), P(a, d), P(d, c)$ and adding $P(e, c)$, $P(c, a)$,$P(a, c)$, and $P(d, e)$ to the relation would make it symmetric.

Answer: Note: The original exam mentioned $P(d, a)$ in answer C. Although irrelevant for the answer (P is not transitive after all), it was not possible to remove $P(d, a)$ as this was not in P to begin with.

A. P is not currently transitive, but adding $P(a, c)$ and $P(b, e)$ is not sufficient, as we need at least also: $P(b, d)$.

B. P is currently reflexive, but we need to remove a tuple (x, x) for some x for it to lose that property.

C. P is not currently transitive, for instance $P(b, d)$ does not hold..

D. This would indeed make the relation symmetric.

Open questions

21. Consider the following two statements written in propositional logic.

    I. $\neg((p \to r) \leftrightarrow q) \vee (p \to q)$

    II. $(p \wedge q) \vee \neg(q \to p)$

    (a) (6 points) Create truth tables for both statements. Clearly indicate with column(s) contain your final answer(s) and please order your rows starting with all zeroes for the propositions.

    Answer: The truth table is as follows:

p	q	r	$p \to r$	$\cdots \leftrightarrow q$	$\neg \ldots$	$p \to q$	$\cdots \vee \ldots$	$p \wedge q$	$(q \to p)$	$\neg(\ldots)$	$\cdots \vee \ldots$
0	0	0	1	0	1	1	1	0	1	0	0
0	0	1	1	0	1	1	1	0	1	0	0
0	1	0	1	1	0	1	1	0	0	1	1
0	1	1	1	1	0	1	1	0	0	1	1
1	0	0	0	1	0	0	0	0	1	0	0
1	0	1	1	0	1	0	1	0	1	0	0
1	1	0	0	0	1	1	1	1	1	0	1
1	1	1	1	1	0	1	1	1	1	0	1

    (b) (1 point) Are the statements equivalent? Explain your answer in at most 3 lines.

    Answer: No, they differ in for instance $p = r = 1$, $q = 0$.

    (c) (4 points) Rewrite statement II to CNF, simplifying the result as much as possible.

    Answer:

$$\begin{aligned}(p \wedge q) \vee \neg(q \to p) &\equiv (p \wedge q) \vee \neg(\neg q \vee p)\\ &\equiv (p \wedge q) \vee (q \wedge \neg p)\\ &\equiv (p \vee q) \wedge (p \vee \neg p) \wedge (q \vee q) \wedge (q \vee \neg p)\\ &\equiv (p \vee q) \wedge q \wedge (q \vee \neg p)\\ &\equiv q\end{aligned}$$

    Alternatively, note that q occurs in both parts of the disjunction on line 2, so at that point the statement can already be simplified to $q \wedge (p \vee \neg p) \equiv q$.

22. Consider the following properties of the integers 0 up to and including 10. Such a number has the property E if it is even. It has the property P if it is prime. It has the property S if it is a square of an integer. Finally, two such numbers x and y have relation R if $x \cdot y > 72$.

    (a) (6 points) Describe the structure above formally. Give the domain as well as the truth sets for the predicates corresponding to the properties and relation above in set-notation.

    Answer: Take the structure A:

    - $D^A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
    - $E^A = \{0, 2, 4, 6, 8, 10\}$
    - $P^A = \{2, 3, 5, 7\}$
    - $S^A = \{0, 1, 4, 9\}$
    - $R^A = \{(8, 10), (9, 9), (9, 10), (10, 8), (10, 9), (10, 10)\}$

(b) (2 points) Consider now the following extra information for this structure:
$\forall x((P(x) \wedge E(x)) \rightarrow Z(x))$
May we now conclude that $\exists x(Z(x))$ for this structure? Explain your answer in at most 5 lines.

Answer: There is an x, $x = 2$, for which it holds that $P(2) \wedge E(2)$ holds. Thus $Z(2)$ holds, thus there is indeed such an x.

23. (a) (6 points) Prove the following claim $\forall n \geq 1, n \in \mathbb{N} : \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.

Answer:

Proof. Proof by induction.
Base case ($n = 1$): $\sum_{i=1}^{1} i(i+1) = 1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$
Inductive step: Assume the claim holds for an arbitrary $k \geq 1$, that is: $\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$ (IH).
To prove: $\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)(k+2)(k+3)}{3}$.

$$\sum_{i=1}^{k+1} i(i+1) = (k+1)(k+2) + \sum_{i=1}^{k} i(i+1)$$

By IH:

$$\begin{aligned} &= (k+1)(k+2) + \frac{k(k+1)(k+2)}{3} \\ &= \frac{3(k+1)(k+2) + k(k+1)(k+2)}{3} \\ &= \frac{(k+1))(k+2)(k+3)}{3} \end{aligned}$$

Since k was arbitrarily chosen it holds for all integers ≥ 1. Thus by induction we have shown that the claim holds. QED

(b) Consider the following algorithm:

function F(n)
  $x \leftarrow 0$
  $c \leftarrow 0$
  while $c \leq n$ **do**
    $c \leftarrow c + 1$
    $x \leftarrow x + c$
  end while
  return x
end function

Consider the invariant $x = \sum_{i=0}^{c} i$.

i. (1 point) Prove that the invariant holds before the loop starts.

Answer: Before the loop: $x = 0 = \sum_{i=0}^{c} i$ as $c = 0$ also.

ii. (3 points) Prove that the invariant holds after an iteration, assuming it holds before the loop.

Answer: Note: This question was removed from the exam scoring as there are two interpretations of the question. Some students have proven that it holds after iteration 1 and others have outlined a proof like the one below to show that it holds after an arbitrary iteration. Due to the large difference in difficulty, the question has been removed.

Proof. After one iteration of the loop: Assume that it holds before the loop starts, that is: $x_{\text{old}} = \sum_{i=0}^{c_{\text{old}}} i$. $c_{\text{new}} = c_{\text{old}} + 1$.

To prove: $x_{\text{new}} = \sum_{i=0}^{c_{\text{new}}} i$.

$$\begin{aligned} x_{\text{new}} &= x_{\text{old}} + c_{\text{new}} \\ &= \sum_{i=0}^{c_{\text{old}}} i + (c_{\text{old}} + 1) \\ &= \sum_{i=0}^{c_{\text{old}}+1} i \\ &= \sum_{i=0}^{c_{\text{new}}} i \end{aligned}$$

QED

iii. (1 point) Explain why the algorithm terminates. Answer in at most 3 lines.

Answer: Eventual falsity of the guard: The variable c increases every iteration, so at some point it will be larger than n which will terminate the loop.

24. (11 points) Below are two claims over sets. One of the claims is true and a proof for that claim can earn you 6 points. One of the claims is false and a counterexample including clear explanation for that claim can earn you 4 points. Correctly indicating which one is true and which one is false gets you an additional 1 point, for a total of 11 points for this question. Indicate clearly which claim you believe to be true and which one false.

(a) For three sets A, B, C: $(A \Delta B) \cap C = (A \cap C)\Delta(B \cap C)$

Answer:

Proof. Consider the following direct proof:

$$\begin{aligned}
(A\Delta B) \cap C &= \{x \mid (x \in A\Delta B) \vee x \in C\} \\
&= \{x \mid ((x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)) \wedge (x \in C)\} \\
&= \{x \mid (x \in A \wedge x \notin B \wedge x \in C) \vee (x \notin A \wedge x \in B \wedge x \in C)\} \\
&= \{x \mid ((x \in A \wedge x \in C \wedge (x \notin B \vee x \notin C)) \vee ((x \notin A \vee x \notin C) \wedge x \in B \wedge x \in C))\} \\
&= \{x \mid (x \in A \cap C \wedge x \notin B \cap C) \vee (x \notin A \cap C \wedge x \in B \cap C)\} \\
&= \{x \mid x \in (A \cap C)\Delta(B \cap C)\} \\
&= (A \cap C)\Delta(B \cap C)
\end{aligned}$$

Therefore $(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C)$ QED

An alternative proof where two subset relations are proven is also worth full marks.

(b) For three sets A, B, C: $(A \subset B \wedge B \cap C \neq \emptyset) \rightarrow A \cap C \neq \emptyset$

Answer: Take $A = \{1\}, B = \{1, 2\}, C = \{2\}$. Now the antecedent is true as A is proper subset of B and $B \cap C = \{2\}$, but $A \cap C = \emptyset$.

25. (a) (5 points) Give a recursive definition of the set A that only contains the number 120, and for any numbers in the set, the set also contains:

- all of the factors/divisors of those numbers.
- all of the products of those numbers.

Answer:

I. $120 \in A$.

II. if $x \in A$, then $\forall y(y \mid x \rightarrow y \in A)$.

III. if $x, y \in A$, then $x \cdot y \in A$.

IV. Nothing else is in A other than the numbers constructed with the rules above.

(b) (8 points) Consider again the recursive definition from Question 13. Prove the following claim: Each word in S contains an odd number of a's.

Answer:

Proof. Define $f(x)$ to return the number of a's in x.
To prove: $\forall x \in S(2 \nmid f(x))$.

Base case ($x = a$): $f(a) = 1$ and $2 \nmid 1$.
Inductive step:
Take arbitrary $x, y \in S$ and assume $2 \nmid f(x) \wedge 2 \nmid f(y)$ (IH).
To prove: $2 \nmid f(xi)$, $2 \nmid f(axa)$, $2 \nmid f(ixiyixi)$.
$f(xi) = f(x) = 2m + 1$ by the IH.
$f(axa) = 2 + f(x) = 2 + 2m + 1 = 2(m + 1) + 1$ by the IH.
$f(ixiyixi) = 2f(x) + f(y) = 2(2m + 1) + 2n + 1 = 4m + 2 + 2n + 1 = 2(2m + n + 1) + 1$ by the IH.
So by the principle of induction, it holds that all words in S have an odd number of a's in it. QED

26. (a) (2 points) Describe in your own words the two differences between functions and relations. Answer in at most 5 lines.

Answer: In relations we can map a single object in the domain to multiple objects in the range. This is not allowed in a well-defined function.

(b) For each of the following functions/relations indicate if they have an inverse. If so, give it. If not, explain why not.

i. (1 point) The function/relation $f : \mathbb{Q} \to \mathbb{Q}$, where $f(x) = 18x + 26$.

Answer: $f^{-1}(y) = (y - 26)/18$

ii. (1 point) The function/relation $g : \mathbb{Z} \to \mathbb{R}$, where $g(x) = x^2 + 5x - 12$.

Answer: This function has no inverse as $x = -5$ and $x = 0$ map to the same value. So the function is not bijective. Alternatively note that $g(x) = 1$ has no integer solution for x. Alternatively mention that $|\mathbb{Z}| < |\mathbb{R}|$.

iii. (1 point) The function/relation $h = \{(a, b), (c, d), (e, f), (g, h), (i, j)\}$.

Answer: $h^{-1} = \{(b, a), (d, c), (f, e), (h, g), (j, i)\}$.

iv. (1 point) The function/relation $\ell = \{(1, 1), (2, 2), (3, 3)\}$.

Answer: $\ell^{-1} = \{(1, 1), (2, 2), (3, 3)\}$.

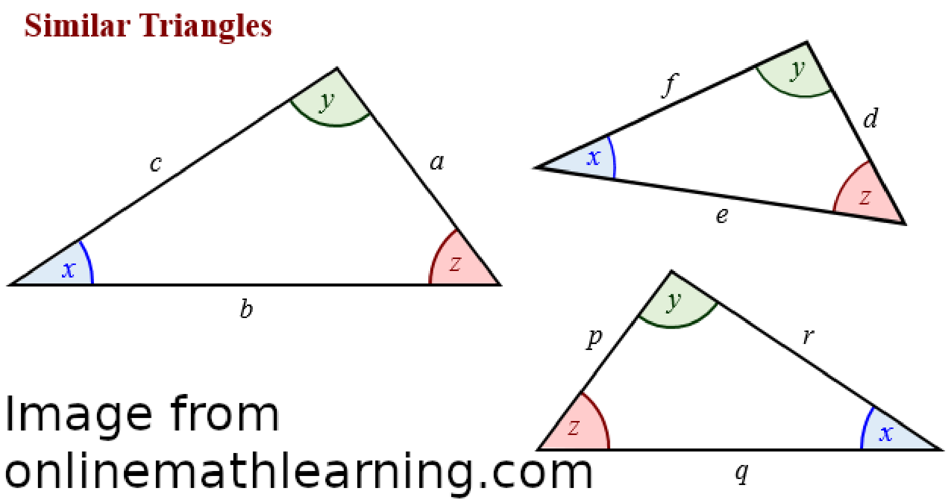


Figure 3: Examples of similar triangles.

(c) (6 points) Out of the following 2 relations, one is an equivalence relation and the other is not. For the relation that is not an equivalence relation, give a counterexample and explanation as to why it is not in at most 5 lines for 2 points. For the relation that is an equivalence relation, explain why it is (discuss *all* properties an equivalence relation should have) in at most 8 lines for 3 points. Correctly indicating which one is false and which one is true gets you 1 point, for a total of 6 points.

i. A relation R over triangles. We say two triangles are *similar* when they have the same set of angles. See Figure 3 for some examples of similar triangles. Let $(x, y) \in R$ for some triangles x, y if and only if x and y are similar.

> **Answer:** This is an equivalence relation. Let a triangle x be represented by a set of three angles $\{x_1, x_2, x_3\}$.
>
> - It it is trivially reflexive as $\{x_1, x_2, x_3\} = \{x_1, x_2, x_3\}$.
> - It is symmetric, as $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\} \leftrightarrow \{y_1, y_2, y_3\} = \{x_1, x_2, x_3\}$ holds.
> - It is transitive, as $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\} \wedge \{y_1, y_2, y_3\} = \{z_1, z_2, z_3\} \rightarrow \{x_1, x_2, x_3\} = \{z_1, z_2, z_3\}$ holds.

ii. A relation B over people. Let a person x have a age a_x and an hourly wage w_x. Then let $(x, y) \in B$ for some people x and y if and only if $w_x + w_y \leq a_x + a_y$.

> **Answer:** The relation is symmetric ($w_y + w_x = w_x + w_y \leq a_x + a_y = a_y + a_x$), but not necessarily reflexive or transitive. Take for instance a person with $a_x = 8, w_x = 12$, then $B(x, x)$ does not hold.

End of the exam