Endterm Reasoning and Logic (CSE1300)

Exam created by Stefan Hugtenburg & Neil Yorke-Smith

Please read the following information carefully!

- This exam consists of 20 multiple-choice questions and 6 open questions.
- The points for the multiple-choice part of the exam are computed as $1 + 9 \cdot \max(0, \frac{\text{score} - 0.25 * 20}{0.75 * 20})$. This accounts for a 25% guessing correction, corresponding to the four-choice questions we use.
- The grade for the open questions is computed as: $1 + 9 \cdot \frac{\text{score}}{64}$.
- The **final grade for the exam** is computed as: $0.4 \cdot MC + 0.6 \cdot Open$.
- This exam corresponds to all chapters of the book: *Delftse Foundations of Computation* (version 1.01).
- You have 3 hours to complete this exam.
- Before you hand in your answers, check that the sheet contains your name and student number, both in the human and computer-readable formats.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Note that the order of the letters next to the boxes on your multiple-choice sheet may **not always be A-B-C-D**!
- Tip: mark your answers on this exam **first**, and only after you are certain of your answers, copy them to the multiple-choice answer form.
- Read every question properly and in the case of the open questions, give **all information** requested, which should always include a brief explanation of your answer. Do not however give irrelevant information – this could lead to a deduction of points.
- Note that the minimum score per (sub)question is 0 points.
- You may write on this exam paper and take it home.
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Open questions:

Question:	21	22	23	24	25	26	Total:
Points:	10	12	8	11	5	18	64

Learning goals coverage, based on the objectives of all lectures (strongly paraphrased):

Goal	mt 17	et 17	mc 18	mt 18	et 18	ret 18
translate logic to and from natural language			1,2	3,4	1	1
describe $\wedge$,$\vee$,$\neg$,$\rightarrow$, and $\leftrightarrow$ operators	1					
construct a truth table	2, 11a		3-5	1a,1b	31a	21a
determine prop. logic equivalence	11b		6,7,19		2	
rewrite logical connectives	6		8-10		31b	21c
describe contrapositives, converses, and inverses.	3		11,12			2
describe logic validity			13,14		3	
describe sufficient and necessary conditions	4		15		4	
prove validty of argument in prop. logic	5		16-17	1b		3, 21b
describe the principle of explosion			18	1c		
explain why prop. logic is not sufficiently expressive			20			
describe $\forall$ and $\exists$ quantifiers	7,8		21	2c	5	
evaluate negation stmt. in pred. logic			22			4
construct a Tarski's world			23-25			
construct a formal structure in pred. logic	12		26-27	2b	32a	22a
evaluate claims about formal structures	12		28-29	2a	6,32b	22b
construct counterexamples for claims	10		30	2a,2b,5c		5
describe the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$						6
describe the form of a proof by division into cases				5b	7	
describe the form of a proof by contradiction						7
construct a proof by division into cases				7a		
construct a proof by contradiction	9					
explain what a theorem prover is.					8	
describe the form of a proof by contrapositive				5a		
construct a proof by contrapositive	13			7b		
describe the form of a proof by generalisation.				5a,5b		
construct a proof by generalisation						
construct an existence proof					9	
identify type of proof to use for a given claim				5b		
compute a sequence given a recursive definition				6a	10	8
construct and interpret recursive definitions		3		6b,6c		
explain the basic principle of an induction proof		2			11	9
construct an induction proof for a claim about numbers					33a	23a
construct an induction proof for algorithms		4			33b	10
construct recursive definitions on sets		12a			12,13	23b, 24
construct a proof using structural induction		12b			14,15	23c
explain and apply basic set operations.		1			16	11
construct Venn diagrams		5			17,18	12
construct formal counterexamples for claims on sets		1,13			19,34b	25
compute the powerset of a set					20,21	13
compute the cartesian product of two sets					22	14
construct proofs for claims on sets					34a	25
describe Cantor's proofs about infinite sets		11b			23	15
construct a function or relation from nat. language					24,25	26a
describe the diff. between a function and a relation					35a	16
determine the inverse of relations and functions		8			35b	26b
determine if a function is well-defined		6			26	17
determine if a function is injective, surjective, or bijective		7,11a,11c			27	18
determine if a relation is symmetric, transitive or reflexive		9			28,29	19
describe the properties of an equivalence relation		10			30	20, 26c

Multiple-Choice questions

1. (1 point) Consider the following statement:
   'Atticus is the only lawyer representing Tom.'

   Which of the following translations to predicate logic is **most** accurate? The predicates used are:

   - a for Atticus.
   - t for Tom.
   - $Lawyer(x)$ for x is a lawyer.
   - $Represents(x, y)$ for x represents y.

   A. $Lawyer(a) \wedge Represents(a, t) \wedge \neg\exists x(Lawyer(x))$
   B. $Represents(a, t) \wedge \forall x((Lawyer(x) \wedge (x \neq a)) \rightarrow \neg Represents(x, t))$
   C. $Lawyer(a) \wedge Represents(a, t) \wedge \exists x(Represents(x, t) \wedge (Lawyer(x) \wedge (x \neq a)))$
   D. $Lawyer(a) \wedge Represents(a, t) \wedge \forall x(Represents(x, t) \rightarrow (\neg Lawyer(x) \vee (x = a)))$

   Answer:
   A. This claims there are no lawyers, but also that Atticus is a lawyer. This is a contradiction.
   B. This does not express that Atticus is a lawyer.
   C. This says there is someone who defends Tom, that is not Atticus.
   D. This says that one lawyer that defends Tom, must be Atticus.

2. (1 point) Which of the following is **true**?

   A. If $p \rightarrow q$ is a tautology, then the converse is guaranteed to be a contradiction.
   B. If $p \rightarrow q$ is a contradiction, then the converse is guaranteed to be a tautology.
   C. If $p \rightarrow q$ is a tautology, then the inverse is guaranteed to be a contradiction.
   D. If $p \rightarrow q$ is a contingency, then the inverse is guaranteed to be a contingency.

   Answer:
   A. The converse is $q \rightarrow p$. This can be true as $p = 0, q = 0$ is allowed for $p \rightarrow q$, but also for $q \rightarrow p$. Therefore $q \rightarrow p$ is not a contradiction.
   B. The converse is $q \rightarrow p$. As $p = 1, q = 0$ is the only allowed state for $p \rightarrow q$ to be a contradiction, which makes $q \rightarrow p$ true, $q \rightarrow p$ must be a tautology.
   C. The inverse is $\neg p \rightarrow \neg q$. This can be true as $p = 0, q = 0$ is allowed for $p \rightarrow q$, but also for $\neg p \rightarrow \neg q$. So $\neq p \rightarrow \neg q$ is not a contradiction.
   D. The inverse is $\neg p \rightarrow \neg q$. If we limit ourselves to $p = 1$, then $p \rightarrow q$ is still a contingency (false for $q = 0$ and true for $q = 1$), however $\neq p \rightarrow \neg q$ is a tautology in that case, meaning $\neg p \rightarrow \neg q$ is not guaranteed to be a contingency.

3. (1 point) Consider the argument: $A, B \therefore C$. Which of the following methods can we use to prove the validity of the argument?

   A. Show that $B \wedge C$ is a contradiction.
   B. Show that $\neg A \wedge C$ is a contradiction.
   C. Show that $B \wedge \neg C$ is a contradiction.
   D. Show that $\neg A \wedge \neg B$ is a contradiction.

Answer: The correct answer needs to either introduce a contradiction within the premises (none of these do that) or we need to make C a premise. The third answer does this, as it shows $\neg B \vee C$ to be true, which combined with premises A and B, shows that C must be true.

4. (1 point) Which of the following formulas is equivalent to: $\neg\forall x\exists y(P(x) \wedge (Q(y) \to \exists z(R(x,y,z))))$?
Note: Carefully read the order of the quantifiers and their associated letters.
   A. $\exists y\forall x\neg(P(x) \wedge (Q(y) \to \exists z(R(x,y,z))))$
   B. $\exists x\forall y(\neg P(x) \vee (Q(y) \wedge \forall z(\neg R(x,y,z))))$
   C. $\exists y\forall x(\neg P(x) \vee (\exists z(\neg R(x,y,z)) \wedge Q(y)))$
   D. $\exists x\forall y(\neg P(x) \vee (\neg Q(y) \to \forall z(\neg R(x,y,z))))$

Answer: Invert both quantifiers, apply DeMorgan, and the negation of an implication and you get to the correct answer.

5. (1 point) Consider the following claims about the positive integers $\mathbb{N}$, where the predicate $P(x)$ is used to indicate that x is prime. Remember that $n \mid m$ is used to represent that n divides m. Which of these claims is true?
   A. $\forall x(((x < 10) \wedge (2 \nmid x)) \to P(x))$
   B. $\forall x((\exists y\exists k((x = yk))) \to \neg P(x))$
   C. $\neg\forall x(\neg P(x) \vee \forall y((y \leq x) \vee (y \nmid x))))$
   D. $\neg\forall x(\neg P(x) \vee \neg\exists y(((2 < y) \wedge (y < x)) \wedge \neg P(y)))$

Answer:
   A. 9 forms a counterexample as an odd number < 10 that is not a prime.
   B. 2 forms a counterexample as it is the multiplication of 2 and 1.
   C. This claim can not be true, as no numbers $y > x$ exists such that $y \mid x$ unless $x = 0$ but 0 is no prime.
   D. 5 is a number that makes this claim true.

6. (1 point) Which of the following statements is **false**?
   A. $\sqrt{16} \in \mathbb{Z}$
   B. $\frac{\pi}{42\pi} \in \mathbb{Q}$
   C. $\log_2(3) \in \mathbb{Q}$
   D. $(\sqrt{-8})^2 \in \mathbb{R}$

Answer:
   A. 4 is an integer.
   B. $\frac{1}{42}$ is a fraction.
   C. $\log_2(3)$ **is irrational.**
   D. -8 is a real number.

7. (1 point) Consider the following proof:

   Proof. Assume $(\neg a \vee \neg b)$ and c to hold. <Derive contradiction> Therefore ... QED

   What may we conclude at the place of the ...?

   A. $(a \wedge b) \rightarrow c$
   B. $c \rightarrow (a \wedge b)$
   C. $(a \wedge b) \rightarrow \neg c$
   D. $\neg c \rightarrow (a \wedge b)$

   Answer: From a proof by contradiction we may conclude: $\neg((\neg a \vee \neg b) \wedge c)$ which is equivalent to answer B.

8. (1 point) Consider the following recursively defined sequence s: $s_0 = 1$, $s_1 = 2$, $s_n = (s_{n-1}+n-1) \cdot s_{n-2}$ for $n \geq 2$.

   What is the value of s_4?

   A. 5
   B. 21
   C. 39
   D. 57

   Answer: $s_2 = (2 + 1) \times 1 = 3$
   $s_3 = (3 + 2) \times 2 = 10$
   $s_4 = (10 + 3) \times 3 = 39$

9. (1 point) Consider a situation where we want to apply induction to prove a property for all integers in $\mathbb{Z}$. What do we now need to do in addition to the regular steps in mathematical induction?

   A. No additional steps are required.
   B. In the base case also show that it holds for $n = -1$.
   C. In the inductive step also show that $\forall n \leq 0(P(n) \rightarrow P(n + 1))$ holds.
   D. In the inductive step also show that $\forall n \in \mathbb{Z}(P(n) \rightarrow P(n - 1))$ holds.

   Answer: We need to send the property downwards as well. Answer B only proves it also holds for -1, answer C does not work either as we have no starting case to send this to for instance $n = -5$. Answer D does exactly what we want though.

10. (1 point) When proving a property of an algorithm with a simple while-loop using induction, we pick a certain invariant. Which of these do we **not** need to show?

    A. Show that the loop runs at least once.
    B. Show that the invariant holds before the loop.
    C. Show that the invariant holds at the end of the code.
    D. Show that if the invariant holds before the loop, it also holds after the end of the loop.

    Answer: It is not necessary for the loop to run at least once for us to prove a property about it. This is what the invariant before the loop part is far.

11. (1 point) Consider the following statements about three arbitrary non-empty finite sets A, B, and C. Which of the following is **guaranteed** to be **true**?

    A. If $A \subseteq B - C$, then $A \subseteq B \vee A \subseteq C$.

    B. If $A \subseteq B \Delta C$, then $A \subseteq B \vee A \subseteq C$.

    C. If $A \subseteq B \cup C^c$, then $A \subseteq B \vee A \subseteq C$.

    D. If $A \subseteq B^c \cap C^c$, then $A \subseteq B \vee A \subseteq C$.

Answer:

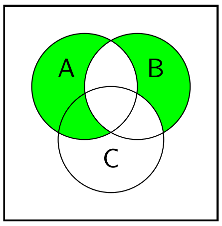
A. True, this means $A \subseteq B$ must hold.

B. False, A can have elements from both $B - C$ and $C - B$.

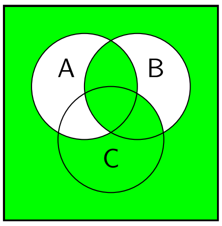
C. False, A can have elements from both B and $B^c \cap C^c$.

D. False, in fact $A \cap B = A \cap C = \emptyset$.

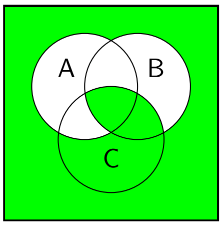
12. (1 point) Let A, B, C be finite non-empty sets. Take $D = ((A - B)\Delta(B - C))^c$. Which of the following diagrams represents this set?



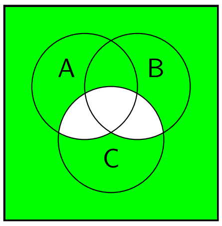
Answer A



Answer B



Answer C



Answer D

Answer:

13. (1 point) Let A, B be arbitrary sets. What can we say with certainty?

    A. If $A \in \mathscr{P}(B)$, then $A \in B$.

    B. If $A \subseteq B$, then $A \in \mathscr{P}(B)$.

    C. If $A = \mathscr{P}(B)$, then $A \subseteq B$.

    D. If $A \in B$, then $\{A\} \subseteq \mathscr{P}(B)$.

Answer: This is the definition of $\mathscr{P}$.

14. (1 point) Consider the sets: $A = \{$Lyra,Jack,Lenny$\}$ and $B = \{$Jethro,Pan,Will$\}$. Which of the following is an element of $A \times B$?

    A. {Will,Lyra}

    B. (Lyra,Jethro)

    C. {(Will, Jack)}

    D. ({Lenny}, {Pan})

Answer: The elements of $A \times B$ are tuples. So just based on that we can remove A and C. Furthermore they are tuples of elements from A and B, so that leaves us with answer B.

15. (1 point) Which of the following claims is **true**?
    - A. $\mathbb{N}$ has a strictly smaller cardinality than $\{\pi^k \mid k \in \mathbb{Q}\}$.
    - **B. $\mathbb{N}$ has the same cardinality as $\{x \in \mathbb{R} \mid x = 2k$ for some $k \in \mathbb{N}\}$.**
    - C. $\mathbb{N}$ has a strictly bigger cardinality than $\{x \mid x = 42k$ for some $k \in \mathbb{N}\}$.
    - D. $\mathbb{Q}$ has a strictly bigger cardinality than $\{x \mid x = 2k + 1$ for some $k \in \mathbb{N}\}$.

Answer: B just describes the even integers, which has the same cardinality as $\mathbb{N}$. All other sets listed here also have the same cardinality as $\mathbb{N}$.

16. (1 point) Consider the following description of a function f. f takes a function g and an integer and returns a fraction. g is a function that takes a real number and an integer and returns a fraction and an integer. Which of the following describes the function f formally?
    - **A. $f : (\mathbb{Q} \times \mathbb{N})^{\mathbb{R}\times\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$.**
    - B. $f : (\mathbb{R} \times \mathbb{N})^{\mathbb{Q}\times\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$.
    - C. $f : (\mathbb{Q})^{\mathbb{R}\times\mathbb{N}} \times \mathbb{N} \to \mathbb{Q} \times \mathbb{N}$.
    - D. $f : (\mathbb{Q})^{\mathbb{Q}\times\mathbb{N}} \times \mathbb{N} \to \mathbb{R} \times \mathbb{N}$.

Answer:

17. (1 point) Which of the following is **true**? *Note:* for this question you should assume that every x and y actually read: $x \in X$ and $y \in Y$.
    - A. In a relation $R \subseteq X \times Y$ there can be only one x for every y.
    - B. In a relation $R \subseteq X \times Y$ there can be only one y for every x.
    - C. In a well-defined function $f : X \to Y$, there can be only one x for every y.
    - **D. In a well-defined function $f : X \to Y$, there can be only one y for every x.**

Answer: This is one of the requirements on well-definedness for functions.

18. (1 point) Consider the following function $L : \mathbb{N} \to \mathbb{Z}$, with $L(n) = \begin{cases} -\dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{else} \end{cases}$.

    Which of the following statements is **true**?
    - A. L is only surjective and not injective.
    - B. L is only injective and not surjective.
    - **C. L has an inverse.**
    - D. L is not well-defined.

Answer: Since L is a bijection from $\mathbb{N}$ to $\mathbb{Z}$, it also has to have an inverse.

19. (1 point) Now that Lyra has arrived in the kingdom of the Panserbjørn[1], she discusses the organisation of the bears with the king. The king indicates that for the sake of the kingdom it would be good to define a relation $C \subseteq B \times B$ where B denotes the set of all bears. We use the relation to express that a certain bear a would like to work with a certain bear b. The king says that ideally the relation is both symmetric and transitive. In that case, which of the following should be **true**?

    A. For every pair of bears: if bear x cannot cooperate with bear y, then y can cooperate with bear x.

    B. Every bear should be able to cooperate with a specific bear x and bear x should be able to cooperate with everyone.

    C. For every bear x that can cooperate with a bear y there should be a bear z such that both x and y can cooperate with z.

    D. For every bear x that cannot cooperate with a bear y, there is no bear z such that x can cooperate with z and z can cooperate with y.

Answer: Answer D is just the contrapositive of the implication that we have in the transitivity property.

20. (1 point) Lyra argues that although the bear king is wise and smart, he should instead aim to find an equivalence relation to divide the bears into groups of bears that can cooperate. After some time Lyra manages to find such a relation and partitions the bears into hunting teams. Which of the following is now **impossible**:

    A. There is only a single hunting team.

    B. Every hunting team is of the same size.

    C. There is a bear that is not part of any team.

    D. There is a hunting team comprised of only one bear.

Answer:

[1] See endterm Reasoning & Logic 2018

Open questions

21. Consider the following two statements written in propositional logic.

I. $(\neg p \vee \neg q) \rightarrow p$

II. $r \rightarrow (p \vee (q \wedge (r \rightarrow \neg q)))$

(a) (6 points) Create truth tables for both statements. Clearly indicate the column(s) that contain your final answer.

Answer: The truth table is as follows:

p	q	r	$\neg p \vee \neg q$	$\cdots \rightarrow p$	$r \rightarrow \neg q$	$q \wedge \ldots$	$p \vee \ldots$	$r \rightarrow \ldots$
0	0	0	1	0	1	0	0	1
0	0	1	1	0	1	0	0	0
0	1	0	1	0	1	1	1	1
0	1	1	1	0	0	0	0	0
1	0	0	1	1	1	0	1	1
1	0	1	1	1	1	0	1	1
1	1	0	0	1	1	1	1	1
1	1	1	0	1	0	0	1	1

(b) (1 point) Consider now the argument: $\dfrac{\text{I}}{\therefore \text{II}}$ where I and II refer to the full statements above. Is this argument valid? If so, explain why. If not, indicate clearly how your truth table shows this (give a concrete counterexample!)

Answer: It is valid. In all cases where the premises are true (when $p = 1$), the conclusion is also true.

Grading rubric:

- 1pt for stating it is valid and that conclusion (II) is true in all cases ($p = 1$) where the premises (I) are true.

(c) (3 points) Rewrite statement II to a form that uses only $\neg$ and $\rightarrow$.

Answer:

$$\begin{aligned} r \rightarrow (p \vee (q \wedge (r \rightarrow \neg q))) &\equiv r \rightarrow (\neg p \rightarrow (q \wedge (r \rightarrow \neg q))) \\ &\equiv r \rightarrow (\neg p \rightarrow \neg (q \rightarrow \neg (r \rightarrow \neg q))) \end{aligned}$$

Grading rubric:

- 1pt for converting the $\vee$ properly.
- 2pt for converting the $\wedge$ properly (can be done in two steps, by applying DeMorgan first).

22. (a) For each of the following descriptions of a function, give a well-defined function that matches the description.

i. (1 point) A function $f : \mathbb{Z} \rightarrow \mathbb{N}$

Answer: For example: $f(x) = |x|$ or $f(x) = 1$.

ii. (2 points) A function $g : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Q}$

Answer: For example: $g(x, y) = (y, x)$ or $g(x, y) = (1, 2)$

iii. (1 point) A function $h : \mathbb{N} \to \{x \mid x = 7k \text{ for some } k \in \mathbb{N}\}$ that is bijective.

Answer: For example: $h(x) = 7x$.

(b) For each of the following *functions/relations*, give their inverse if they have one. If they do not have an inverse, give a clear and concrete example that shows *why* the function/relation does not have an inverse.

i. (1 point) $f : \mathbb{Q} \to \mathbb{Z}$ where $f(x) = -6x + 17$.

Answer: This function is not well-defined, so has no inverse. As $f(-0.25) = 19.5 \notin \mathbb{Z}$.

ii. (1 point) $g : \mathbb{N} \to \mathbb{N}$ where $g(x) = x^2$.

Answer: This function is not surjective, so it has no inverse. There is no integer x such that $g(x) = 2$.

iii. (1 point) $h : \{1, 2, 3\} \to \mathbb{N}$ where $h = \{(1, 2), (2, 3), (3, 2)\}$

Answer: This has no inverse, as h is not injective. $h(1) = h(3)$.

iv. (1 point) $l \subseteq \mathbb{N} \times \mathbb{N}$, with $l = \{(1, 2), (2, 3), (3, 2)\}$

Answer: Take $l^{-1} = \{(2, 1), (3, 2), (2, 3)\}$

(c) (4 points) Out of the following 2 relations, one is an equivalence relation and the other is not. For the relation that is not an equivalence relation, give a counterexample and explanation as to why it is not in at most 5 lines for 1 point. For the relation that is an equivalence relation, explain why it is (discuss *all* properties an equivalence relation should have) in at most 8 lines for 2 points. Correctly indicating which one is an equivalence relation and which one is is not gets you 1 point, for a total of 4 points.

i. The relation P between snails defined as follows: A snail a and a snail b are in the relation P iff they share at least one parent.

Answer: Take a snail a with *Alexander* and d as its parents.
b with *Alexander* and *Gerrit* as its parents.
c with e and *Gerrit* as its parents.
Now even though both a, b and b, c are in the relation, a, c is not. So the relation is not transitive, thus not an equivalence relation.

ii. The relation T between snails defined as follows: A snail a and a snail b are in the relation T iff they share a terrarium (a housing unit for snails).

Answer:

- Every snail is in the same terrarium as itself (reflexive).
- If a is in the same terrarium as b, then the reverse also holds (symmetric).
- If a is in the same terrarium as b and b is in the same one as c, then a and c are also in the same terrarium (transitive).

23. Construct **recursive** definitions for the two sets S and T below:

(a) (4 points) The set $S \subseteq \mathbb{N}$ contains the numbers 6 and 7. Furthermore any sum or product of two numbers in S is also in S. Finally any number that is 5 away from any number in S on a number line is also in S. (For example both 2 and 12 are in S as they are 5 away from 7.)

Answer:

I. $6, 7 \in S$

II. $\forall x, y : x, y \in S \rightarrow x + y \in S$

III. $\forall x, y : x, y \in S \rightarrow xy \in S$

IV. $\forall x, y : (x \in S \wedge |x - y| = 5) \rightarrow y \in S$

V. (OPTIONAL!) Nothing else is in S.

(b) (4 points) The set T contains the fraction $\frac{1}{2}$. Furthermore every number t in the set can be written as: $t = \frac{2^k}{2^l}$ where $k, l \in \mathbb{N}$.

Answer:

I. $\frac{1}{2} \in T$

II. $\forall x : x \in T \rightarrow 2x \in T$

III. $\forall x : x \in T \rightarrow x/2 \in T$

IV. Nothing else is in T.

24. (11 points) Below are two claims over sets. One of the claims is true and a proof for that claim can earn you 6 points. One of the claims is false and a counterexample including clear explanation for that claim can earn you 4 points. Correctly indicating which one is true and which one is false gets you an additional 1 point, for a total of 11 points for this question. Indicate clearly which claim you believe to be true and which one false.

(a) For three sets A, B, C: $(A \subseteq B \wedge (B \cup C) \subseteq A) \rightarrow A = B$

Answer:

Proof. Take three sets A, B, C such that $(A \subseteq B \wedge (B \cup C) \subseteq A)$.
We now prove two things:

- Take an arbitrary element $x \in A$. To prove: $x \in B$.
  $x \in A$ means that $x \in B$ (from $A \subseteq B$).
  Thus $A \subseteq B$.
- Take an arbitrary element $x \in B$. To prove: $x \in A$.
  $x \in B$ means that $x \in (B \cup C)$, so $x \in A$ (from $(B \cup C) \subseteq A$).
  Thus $B \subseteq A$.

Since $A \subseteq B \wedge B \subseteq A$, it follows that $A = B$. QED

Grading rubric:

- Correct proof technique for $A = B$: 1pt.
- Correctly taking arbitrary elements: 1pt.
- Correct proof for $A \subseteq B$: 1pt.
- Correct proof for $B \subseteq A$: 2pt.
- Conclusion: 1pt.

(b) For three **non-empty** sets A, B, C within some universe U: $(B - C)^c \cap (A \cup B) \subseteq A$

> **Answer:** Take $U = \{1, 2\}, A = \{1\}, B = C = \{2\}$. Now $(B - C)^c = \{1, 2\}$, $A \cup B = \{1, 2\}$ so $(B - C)^c \cap (A \cup B) = \{1, 2\}$ which contains $2 \notin A$.

25. Consider the following set of words W: {snail, alexander, anay, phoenix, nick, maya, pearl, gerrit}. We define the following predicates over W. A is the set of all words that contain the letter a. B is the set of all words contain the same letter more than once. All pairs of words of length 4 and a word of length 8 that start with the same letter form the relation R.

(a) (3 points) Describe the structure above formally. Give the predicates corresponding to the properties and relation above in set-notation.

> **Answer:** Take the structure S:
>
> - $A^S =$ {snail, alexander, anay, maya, pearl}
> - $B^S =$ {alexander, anay, maya, gerrit}
> - $R^S = \{\}$ (There are no words of length 8)
>
> **Grading rubric:**
>
> - 1 pt each for A, B, R.

(b) (1 point) Consider now the following extra information for this structure:
$\forall x((A(x) \land B(x)) \rightarrow Z(x))$
A student claims that we can read the predicate Z in our structure as: Z is the set of all words that contain at least two a's. Is the student correct? If so, explain why. If not, give a counterexample.

> **Answer:** The student is correct. In this all words both in A and in B (alexander, anay, maya) contain two a's.
> **Grading rubric:**
>
> - 1pt for stating something about the intersection of A and B.

(c) (1 point) Consider now that we add the word *harry* to the set W. Is the translation of Z for this new set W correct? If so, explain why. If not, give a counterexample.

> **Answer:** It now no longer holds. *harry* is both in A and B, but does not have two as.
> **Grading rubric:**
>
> - 1pt for stating that the words is in both A and B, but does not have two as.

26. (a) (8 points) Consider the sequence: $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for all $n > 2$. Prove that $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ for all $n \geq 1$. *Hint:* use strong induction for your proof.

> **Answer:**
>
> *Proof.* Proof by **strong** induction.
> Base cases:
> $n = 1$: $a_1 = 3 \cdot 2^{1-1} + 2 \cdot (-1)^1 = 3 - 2 = 1$
> $n = 2$: $a_2 = 3 \cdot 2^{2-1} + 2 \cdot (-1)^2 = 6 + 2 = 8$
> Inductive step:
> Assume the claim holds for all values $\leq$ some arbitrary $k \geq 2$, that is: $a_m = 3 \cdot 2^{m-1} + 2 \cdot (-1)^m$ for all $m \leq k$ (IH).

To prove: $a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^{k+1}$.

$$a_{k+1} = a_k + 2a_{k-1}$$

By IH:

$$\begin{aligned}
&= 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 2(3 \cdot 2^{k-2} + 2 \cdot (-1)^{k-1}) \\
&= 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2^{k-1} + 4 \cdot (-1)^{k-1} \\
&= 2(3 \cdot 2^{k-1}) + 2 \cdot (-1)^k - 4 \cdot (-1)^k \\
&= 3 \cdot 2^k + 2 \cdot -1 \cdot (-1)^k \\
&= 3 \cdot 2^k + 2 \cdot (-1)^{k+1}
\end{aligned}$$

Since k was arbitrarily chosen it holds for all integers ≥ 1. Thus by induction we have shown that the claim holds. QED

(b) (2 points) Consider the following recursively defined set S:

I. $\pi \in S$
II. $\forall x : x \in S \to \pi + x \in S$
III. $\forall x, y : x, y \in S \to \frac{x}{y} \in S$
IV. Nothing else is in S.

Explain why $\{x \in \mathbb{N} \mid x > 0\} \subseteq S$. Be clear what rules should be applied to form the elements from $\{x \in \mathbb{N} \mid x > 0\}$.

Answer: Applying rule II gets us $\{\pi, 2\pi, 3\pi, \dots\}$. We can then apply rule III and divide each of these elements by π to get all elements of $\mathbb{N}$.

(c) (8 points) Now consider the following recursively defined set T:

I. $8 \in T$
II. $\forall x : x \in T \to 7x \in T$
III. $\forall x : x \in T \to 3x^2 + 2x + 18 \in T$
IV. $\forall x, y : x, y \in T \to x^y \in T$
V. Nothing else is in T.

Prove that every element in T is even.

Answer:

Proof. Proof by **structural** induction.
Base cases (8):
$8 = 2 \cdot 4 = 2c$ for $c = 4$. Thus the base case holds.
Inductive step:
Assume the claim holds for some arbitrary $k \in T$, that is $k = 2c$ for some integer c. We now need to prove the three rules do not break the property.

- $7k = 7 \cdot 2c = 2 \cdot 7c = 2c_1$ for $c_1 = 7c$.
- $3k^2 + 2k + 18 = 3 \cdot (2c)^2 + 2 \cdot 2c + 18 = 3 \cdot 4c^2 + 4c + 18 = 2(6c^2 + 2c + 9) = 2c_1$ with $c_1 = 6c^2 + 2c + 9$.

- Further assume that for $l \in T$ it also holds that $l = 2c_1$. $k^l = (2c)^{2c_1} = 2^{2c_1}c^{2c_1} = 2 \cdot (2^{2c-1}c^{2c_1}) = 2c_2$ with $c_2 = 2^{2c-1}c^{2c_1}$. This is valid as $l \geq 8 > 0$ (numbers only increase by the rules, not decrease).

Since k was arbitrarily chosen it holds for all integers $\in T$. According to rule V there is nothing else in T, so by induction we have shown that the claim holds. QED

End of the exam