NOTE: This exam was never conducted on paper, but digitally! As a result some questions have been formulated slightly differently, the layout of this PDF is not optimised, and other issues may appear that were not present in the digital version of this exam.

Additionally the midterm was formative in the year of this endterm, meaning more questions were asked here about the midterm material.

1. (5 points) • **Variant Kokofruit:**
   For this question you need to translate claims from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.
   (a) – **Variant 2**For this question you need to translate a Kokofruit grows on tall kokotrees from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.
   Kokofruit grows on tall kokotrees

   Answer: $\forall x(Kokofruit(x) \rightarrow \exists y(Tall(y) \wedge Kokotree(y) \wedge GrowsOn(x, y)))$
   Grading rubric:
   * 1pt for the predicates/constants.
   * 1pt for the quantifiers and the correct connectives in the correct place.

   (b) – **Variant 1**For this question you need to translate a If you save some trunk, you have a floor. from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.
   If you save some trunk, you have a floor.

   Answer: $\exists x(Trunk(x) \wedge Save(you, x)) \rightarrow \exists y(Floor(y) \wedge Has(you, y))$
   Grading rubric:
   * 1pt for correct predicates/constants.
   * 1pt for correct quantifiers and connectives.
   * 1pt for correctly scoping the quantified statements

2. (4 points) • **Variant WILTY:**
   For this question you need to translate claims from predicate logic to natural language. We use the following predicates and constants:
   – d is for David
   – l is for Lee
   – v is for Victoria
   – $Panelist(x)$ for x is a Panelist
   – $Lollypopman(x)$ for x is a lollypop man (someone who helps children cross the street)
   – $Accuses(x, y)$ for x accuses y of lying
   – $IsCaptainOf(x, y, z)$ for x is captain of team with other members y and z

   These predicates and constants are inspired by the tv-show Would I lie to you?
   (a) – **Variant 2**For this question you need to translate a claim from predicate logic to natural language. We use the following predicates and constants:
   * d is for David
   * l is for Lee
   * v is for Victoria
   * $Panelist(x)$ for x is a Panelist
   * $Lollypopman(x)$ for x is a lollypop man (someone who helps children cross the street)
   * $Accuses(x, y)$ for x accuses y of lying
   * $IsCaptainOf(x, y, z)$ for x is captain of team with other members y and z

   These predicates and constants are inspired by the tv-show Would I lie to you?
   The claim: $\exists x(Lollypopman(x) \wedge x \neq d) \rightarrow Panellist(l)$

   Answer: If there is a lollypop man that is not David, then Lee is a panellist.
   Grading rubric:
   * 1pt for the part with the quantifier.
   * 1pt for correct overall implication.

(b) – **Variant 0**For this question you need to translate a claim from predicate logic to natural language. We use the following predicates and constants:
* d is for David
* l is for Lee
* v is for Victoria
* $Panelist(x)$ for x is a Panelist
* $Lollypopman(x)$ for x is a lollypop man (someone who helps children cross the street)
* $Accuses(x, y)$ for x accuses y of lying
* $IsCaptainOf(x, y, z)$ for x is captain of team with other members y and z

These predicates and constants are inspired by the tv-show Would I lie to you?
The claim: $\forall x \forall y \forall z(Captain(x, y, z) \rightarrow (x = d \lor x = l))$

Answer: Only David and Lee are team captains.
Grading rubric:
* 1pt for correct order of the implication.
* 1pt for correctly reading the ternary operator

3. (7 points) Note that the arguments look a bit weird in the PDF, this is so they look better in WebLab (hopefully)

1. **Variant 7:**
Consider the following quaternary operator $p \xrightarrow[r]{q} s$, with the truth table:

p	q	r	s	$p \xrightarrow[r]{q} s$
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Consider now the following argument:

$$\frac{r \xrightarrow[(p \lor \neg q)]{r \leftrightarrow p} q}{\therefore p \xrightarrow[p]{q} q}$$

- For 6 points create a single truth table containing the truth values of the premise and of the conclusion.
- For 1 point explain how your truth table shows the argument is (in)valid, refer to specific lines from your truth table in your explanation.

You may write the quaternary operator as p - (q,r) -> s or in LaTeX.

For your convenience, here is also a version of the argument in LaTeX (please note that in TeX the square brackets for the xrightarrow indicate what goes below, and the curly braces what goes above the arrow):

```
\\(r \xrightarrow[(p \vee \neg q)]{r \leftrightarrow p} q\\)
\\(\therefore p \xrightarrow[p]{q} q\\)
```

For a slightly larger table, you can copy/paste this. Please note that this table might be too big or still too small! You may remove unused columns/rows or simply leave them empty.

```
| col 1  | col 2 | col 3 | col 4 | col 5 | col 6 | col 7 | col 8 | col 9 | col 10 | col 11 |
|:------:|:-----:|:-----:|:-----:|:-----:|:-----:|:-----:|:-----:|:-----:|:------:|:------:|:
| row 1  |       |       |       |       |       |       |       |       |        |        |
| row 2  |       |       |       |       |       |       |       |       |        |        |
| row 3  |       |       |       |       |       |       |       |       |        |        |
| row 4  |       |       |       |       |       |       |       |       |        |        |
| row 4  |       |       |       |       |       |       |       |       |        |        |
| row 5  |       |       |       |       |       |       |       |       |        |        |
| row 6  |       |       |       |       |       |       |       |       |        |        |
| row 7  |       |       |       |       |       |       |       |       |        |        |
| row 8  |       |       |       |       |       |       |       |       |        |        |
| row 9  |       |       |       |       |       |       |       |       |        |        |
| row 10 |       |       |       |       |       |       |       |       |        |        |
| row 11 |       |       |       |       |       |       |       |       |        |        |
| row 12 |       |       |       |       |       |       |       |       |        |        |
| row 13 |       |       |       |       |       |       |       |       |        |        |
| row 14 |       |       |       |       |       |       |       |       |        |        |
| row 15 |       |       |       |       |       |       |       |       |        |        |
| row 16 |       |       |       |       |       |       |       |       |        |        |
```

Answer:

p	q	r	$\overbrace{p \vee \neg q}^{Z}$	$\overbrace{r \leftrightarrow p}^{A}$	$r \xrightarrow[Z]{A} q$	$p \xrightarrow[p]{q} q$
0	0	0	1	1	0	0
0	0	1	1	0	0	0
0	1	0	0	1	0	0
0	1	1	0	0	1	0
1	0	0	1	0	1	0
1	0	1	1	1	0	0
1	1	0	1	0	0	1
1	1	1	1	1	1	1

Cleary the argument is invalid, as shown by row 4 which makes the premise true, but the conclusion false. **Grading rubric:**

- 1pt for doing $\vee$ correctly.
- 1pt for doing $\leftrightarrow$ correctly.
- 2pt for doing the quatenary operator correctly at least once.
- 2pt for doing the quatenary operator correctly both times.
- 1pt for highlighting a line that shows the invalidity of the argument and mentioning it makes the premise true, but the conclusion false.

4. (2 points) • **Variant 6:**

For one point give a valid argument whose conclusion is: "there are no surjective functions".
We say an argument is sound if it is valid and if its premises are also true. For one point explain whether or not your argument is sound.

Answer: $1 = 2$

$\therefore$ there are no surjective functions

Clearly this argument is not sound as $1 \neq 2$, but because of this contradiction in the premises the argument is valid (principle of explosion). **Grading rubric:**

- 1pt for a valid argument (likely based on the principle of explosion)
- 1pt for explaining correctly why it is not sound.

5. (a) (3 points) • **Var7**

If the following set of statements is satisfiable prove it with a structure with at least 3 and at most 5 elements in the domain. If it is not, explain why not.

- $\forall x \neg(\neg Q(x) \vee P(a))$
- $\exists y \forall z(\neg R(y, z) \rightarrow Q(z))$
- $(Q(c) \vee R(a, b))$

Answer: Take the structure $D = \{a, c, b\}, Q = \{a, c, b\}, P = \{\}$, $R = \{(a, b)\}$.

Grading rubric:

- 1pt for satisfying the unquantified statement.
- 1pt for satisfying the singly quantified statement
- 1pt for satisfying the doubly quantified statement
- -1pt for missing domain or missing constants in the domain.

(b) (2 points) • **Var2**

Donna the coati queen and Marty the owl king are trying to resolve their differences, it is time to move on from the tragic events that turned them against each other all those years ago. Unfortunately the inhabitants of their kingdoms have different ideas about this and are not quite ready to bury the hatchet. During the first of a series of what are supposed to be peaceful events, the coatis and owls gathered there have taken up arms and are ready to fight.
The figure below shows these creatures ready to fight each other! (Just to make sure, a is a coati and b is an owl).

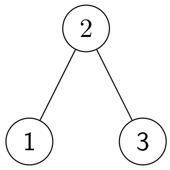
In order to assuage the situation, you try to make sure the following claim is **true**:

$\forall x((Coati(x) \land CrystalBall(x)) \rightarrow \exists y(Owl(y) \land LeftOf(x, y)))$
Is the claim already true? If so, explain why. If it is not, explain which coati or owl we should move to make the claim true and how that makes the claim true. Please note you can only move one owl or one coati!

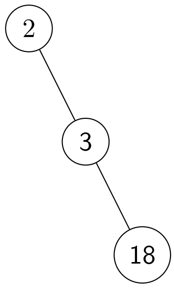
> **Answer:** We can move an owl (any of them) to the right of the right-most coati. Now all coatis are to the left of that owl, so surely also all the coati that are carrying a crystall ball.
> **Grading rubric:**
> - 1pt correct coati/owl moved.
> - 1pt correct explanation of why this makes it true.

6. (4 points) • **Variant 2** Consider the following recursively defined structure T that represents a set of trees as used in computer science.
   - $\emptyset \in T$
   - $(T_1, T_2 \in T \land r \in \mathbb{N}) \rightarrow (T_1, r, T_2) \in T$
   - Nothing else is in T

   You can visualise a tree like $((\emptyset, 1, \emptyset), 2, (\emptyset, 3, \emptyset))$ as follows:



   You can visualise a tree like $(\emptyset, 2, (\emptyset, 3, (\emptyset, 18, \emptyset)))$ as follows:



   Create a recursive function $f : T \rightarrow \mathbb{N}$ such that: f(t) gives the largest of the numbers in t, or 0 if no such value exists

> **Answer:** $f(t) = \begin{cases} 0 & \text{if } t = \emptyset \\ \max(f(T_1), r, f(T_2)) & \text{if } t = (T_1, r, T_2) \end{cases}$ Alternatively we can split this with an if on f(T1) and f(T2)
> **Grading rubric:**
> - 1pt correct base case
> - 1pt correctly recursing on both parts of the tree.
> - 1pt correct if/else statements for the recursive call(s)
> - 1pt correctly combining results of recursive call(s)

7. (7 points) • **Variant 3:**
   Prove the following claim with mathematical induction:
   $\forall n \geq 4, n \in \mathbb{N} \quad \prod_{i=4}^{n} (\frac{i^2}{2} - i) = \frac{n! \cdot (n-2)!}{3 \cdot 2^{n-2}}$
   For your convenience, in LaTeX:

```
$\forall n \geq 4,n \in \mathbb{N} \quad \prod_{i = 4}^{n} (\frac{i^2}{2} - i) = \frac{n! \cd
```

Note that you may structure your equations in a list, something like:

```
- 1 + 1 =
- 1 + 1 - 2 + 2 =
- -1 + 3 =
- 2
```

Answer:

Proof. Proof by induction.
Base case (n = 4):

$$\prod_{i=4}^{4}(\frac{i^2}{2} - i) = (\frac{4^2}{2} - 4) = 4 = \frac{24 \cdot 2}{3 \cdot 4} = \frac{4! \cdot (4-2)!}{3 \cdot 2^{4-2}}$$

Inductive step: Assume the claim holds for any arbitrary $k \geq 3$, that is:

$$\prod_{i=4}^{k}(\frac{i^2}{2} - i) = \frac{k! \cdot (k-2)!}{3 \cdot 2^{k-2}}$$

(IH).
To prove:

$$\prod_{i=4}^{k+1}(\frac{i^2}{2} - i) = \frac{(k+1)! \cdot (k-1)!}{3 \cdot 2^{k-1}}$$

$$\begin{aligned}\prod_{i=4}^{k+1}(\frac{i^2}{2} - i) &= \prod_{i=4}^{k}(\frac{i^2}{2} - i) \cdot (\frac{(k+1)^2}{2} - (k+1)) \\ &= \frac{k! \cdot (k-2)!}{3 \cdot 2^{k-2}} \cdot \frac{(k+1)^2 - 2k - 2}{2} \\ &= \frac{k! \cdot (k-2)!}{3 \cdot 2^{k-2}} \cdot \frac{k^2 - 1}{2} \\ &= \frac{k! \cdot (k-2)!}{3 \cdot 2^{k-2}} \cdot \frac{(k+1)(k-1)}{2} \\ &= \frac{(k+1)! \cdot (k-1)!}{3 \cdot 2^{k-1}}\end{aligned}$$

Since k was arbitrarily chosen it holds for all integers ≥ 4. Thus by induction we have shown that the claim holds. QED

Grading rubric:

- 1pt for a correct base case
- 1pt for stating the inductive hypothesis
- 1pt for rewriting a product of $k+1$ iterations to a product of k iterations
- 1pt for rewriting the factorials of k and $k-2$ to factorials of $k+1$ and $k-1$
- 1pt for applying the inductive hypothesis
- 1pt for the rest of the arithmetic operations
- 1pt for the conclusion that the claim thus holds for all integers at least 4

8. (8 points) • **Var 5:**
Consider the following recursively defined set S of words:

I. $neil, b \in S$
II. $x \in S \rightarrow azxaa \in S$

III. $x, y \in S \rightarrow zxbyba \in S$

IV. Nothing else is in S

For 1 point give an example of a word of length 5 that is in S. (Note that the length of a words is the number of letters in it. For example the length of the word "spoon" is 5.)

For 6 points, prove that $\forall w \in S$(twice the number of z's in w is at most as high as the number of a's)

Answer: A word in S of length 5: azbaa

Proof. Proof by structural induction.

Let $f_z(w)$ denote the number of z's in w and $f_a(w)$ denote the number of a's. The claim now reads $\forall w(2f_z(w) \leq f_a(w))$.

Base case ($w = neil, b$): For the words neil and b: $f_a(neil) = 0, f_z(neil) = 0$, so $f_a(neil) \geq 2 \cdot f_z(neil)$ and $f_a(b) = 0, f_z(b) = 0$, so $f_a(b) \geq 2 \cdot f_z(b)$.

Inductive step: Assume the claim holds for arbitrary words $k, l \in S$, that is: $f_a(k) \geq 2 \cdot f_z(k)$ and $f_a(l) \geq 2 \cdot f_z(l)$

To prove: $f_a(azxaa) \geq 2 \cdot f_z(azxaa)$ and $f_a(zxbyba) \geq 2 \cdot f_z(zxbyaa)$

$$\begin{aligned}
f_a(azkaa) &= f_a(a) + f_a(z) + f_a(k) + 2 \cdot f_a(a) \\
&= 3 + f_a(k) \\
\text{(By IH)} &\geq 3 + 2 \cdot f_z(k) \\
&> 2 + 2 \cdot f_z(k) \\
&= 2(1 + f_z(k)) \\
&= 2(f_z(a) + f_z(z) + f_z(k) + 2 \cdot f_z(a)) \\
&= 2f_z(azkaa)
\end{aligned}$$

$$\begin{aligned}
f_a(zkblba) &= f_a(z) + f_a(x) + f_a(b) + f_a(l) + 2 \cdot f_a(a) \\
&= f_a(k) + f_a(l) + 2 \\
\text{(By IH)} &\geq 2 \cdot f_z(k) + 2 \cdot f_z(l) + 2 \\
&= 2(f_z(k) + f_z(l) + 1) \\
&= 2(f_z(z) + f_z(x) + f_z(b) + f_z(l) + f_z(b) + f_z(a)) \\
&= 2f_z(zkblba)
\end{aligned}$$

Since k and l were arbitrarily chosen it holds for all words that can be produced. Thus by induction we have shown that the claim holds. QED

Grading rubric:

- 1pt for a correct word
- 1pt for a correct base case
- 1pt for stating the inductive hypothesis, including the notion of arbitrary word
- 1pt for applying the inductive hypothesis correctly at least once.
- 1pt for the step that requires the $>$ instead of the $\geq$.
- 1pt for the first recursive case
- 1pt for the second recursive case
- 1pt for the conclusion of the proof

9. • **Variant 9:**
Given the sets $A = \{19, \text{impostor}, \text{marty}, b, d, u, x\}$, $B = \{9, \text{coati}, \text{maya}, b, d, o, u\}$, $C = \{9, \text{impostor}, \text{maya}, o, q, r$
give (the elements of) the set $(A \cup B) - C$.

Answer: $\{19, \text{coati}, \text{marty}, b, d, u, x\}$
Grading rubric:

– 1pt correct answer

(a) (3 points) • **Variant 3:**
Consider the following claim:
For all sets A, B, C: $((A \subseteq B) \wedge (\mathscr{P}(B) \cap C \neq \emptyset)) \to A \neq \emptyset$.
If this claim is true, prove it. If this claim is false, provide a counterexample and explain how your counterexample shows the claim is false.
For your convenience, here is the claim in LaTeX:

```
$((A \subseteq B) \wedge (\mathscr{P}(B) \cap C \neq \emptyset)) \to A \neq \emptyset$
```

Answer: Counterexample: $A = \emptyset, B = \{1\}, \mathscr{P}(B) = \{\emptyset, \{1\}\}, C = \{\{1\}\}$. Now the antecedent is clearly true as $\mathscr{P}(B) \cup C = \{\{1\}\}$ and $A \subseteq C$. However the consequent is false as $A = \emptyset$.
Grading rubric:

– 2pt correct counterexample.

– 1pt explain why the counterexample is correct, which means at least mentioning the antecedent is true and the consequent is false.

10. (3 points) • **Variant 4:**
In a Tarski world, consider the relation $NotBelow(x, y)$ where $NotBelow(x, y)$ is true iff x is not in a lower row than y. Describe if this is an equivalence relation by discussing the three properties of an equivalence relation and for each indicating why it does (not) hold.

Answer:

– It is reflexive, as x and x are in the same row, so x is not in a lower row than x

– It is not symmetric, since if x is above y, then x is not in a lower row than y, but y is in a lower row than x

– It is transitive, as if x is not in a lower row than y, and y is not in a lower row than z, x will also not be in a lower row than z

Grading rubric:

– 1pt for a correct argumention of the reflexive property

– 1pt for a correct argumention of the symmetric property

– 1pt for a correct argumention of the transitive property

11. (4 points) • **Variant 3:**
Given the set of pairs (a, b) consisting of a natural number a and a natural number b which is at least 1 and lower or equal to the $(a + 1)^{th}$ odd natural number, formally given by:
$A = \{(a, b) | a \in \mathbb{N} \wedge b \in \mathbb{N} \wedge 1 \leq b \leq 2 \cdot a + 1\}$
Prove that A is countably infinite by giving a bijection $A \to \mathbb{N}$ and explaining why this function is a bijection.
Hint: remember that the differences between each two subsequent squares are all the odd numbers ($(x + 1)^2 - x^2 = 2x + 1$).

Answer: $f : A \to \mathbb{N}$ is defined by $f(a, b) = a^2 + b - 1$.
f is bijective because for all a, there are $a + 1$ possible values for b, and they map to the $2a + 1$ natural numbers that are at least a^2 and lower than $(a + 1)^2$. This way, all natural numbers are mapped to by exactly 1 pair (a, b).

Grading rubric:

- 2pt for the correct function
- 1pt for correctly explaining it's a surjective function
- 1pt for correctly explaining it's an injective function

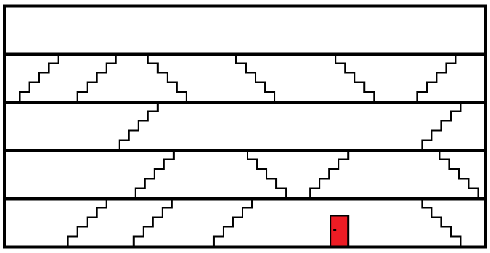
12. (1 point) • **Variant 2** Show the following function is not well-defined:
$f : \mathbb{R} \rightarrow \mathbb{N}$, with $f(x) = x^2$

Answer: Take $f(\pi)$, this has no mapping to a number in $\mathbb{N}$

Grading rubric:

- 1pt correct counterexample

13. (4 points) • **Variant 0:**
In a building with multiple floors, you enter via the entrance at ground floor (the lowest floor). Then, you take a random staircase on that floor that you haven't taken before to another floor. You keep taking random staircases that you haven't taken before (either a staircase on your floor that leads to the floor above, or a staircase from the floor below that leads to the floor below) until you are at a floor where you have already taken all available staircases. It is given that each floor has an even amount of staircases (and the top one has no staircases). An example of a building is given below.



With a proof using an invariant, you can prove at which floor you will end up. The invariant is: each of the floors below you has an odd amount of staircases that you have walked, the floor where you are and each of the floors above you have an even amount of staircases that you have walked.

For 3 points: Show the invariant is an invariant, i.e. it is true before the loop and after every iteration of the loop.

For 1 point: Given this invariant, what floor do we end up on? Explain your answer (a formal proof is not required).

Answer: Before the loop, no staircases were walked yet, so at each floor, an even amount of staircases have been walked. This is true for each floor, so it also true for the floor where you are, and each of the floors above you.

Before the loop, you are at ground floor, so there are no floors below you, which means anything is true for all floors below you.

When you walk down from floor i to floor $i-1$, you don't walk on staircases from floors above or below $i-1$. Since you walked on a respectively even and odd amount of those per floor, that is still true. You did walk on a staircase from floor $i-1$, and since you walked on an odd amount of them before that, you now walked on an even amount of them.

When you walk up from floor i to floor $i+1$, you don't walk on staircases from floors above or below i. Since you walked on a respectively even and odd amount of those per floor, that is still true. You did walk on a staircase from floor i, and since you walked on an even amount of them before that, you no walked on an odd amount of them.

When we stop, there are no available staircases anymore. In particular, that means there are no avaible staircases on the floor below us where we have walked an odd amount of them, or we are at ground floor. Since the first means there are an odd amount of staircases on the floor below us, which is a contradiction, we are at ground floor.

Grading rubric:

- 1pt for a correct argument that the invariant is true before the loop
- 1pt for a correct argument that the invariant is true after you walk down
- 1pt for a correct argument that the invariant is true after you walk up
- 1pt for the correct argumentation for the final answer

14. (3 points) • **Variant 6:**

Four young apprentices broke into a temple to steal four sacred element crystals. When the alarm went off, they panicked, and each of them swallowed the crystal they held right before they were caught. You must determine who ate which crystal. The elements compel their masters. Those who ate the earth and water crystals must speak the truth, while those who consumed fire and air must lie. The youths are too scared to confess their own transgressions. Instead, they fall to accusing each other. This is what they said:

- Donna said something before you arrived (but you don't know what)
- Coyote Caoti says that Coyote Caoti is telling the truth about that statement
- Marty says that she didn't eat the water crystal
- Dr. Whoo says that if Coyote Caoti ate the fire crystal, then Dr. Whoo ate the water crystal

Remember that:

- Each student ate a different elemental crystal
- The earth and water crystals force their owner to tell the truth
- The fire and air crystals force their owners to lie

Who ate what crystal? Explain how you derived your answer.

Answer: Suppose Marty would be lying. Then, he ate the fire crystal or the air crystal, so he didn't eat the water crystal. But then Marty is telling the truth, which is a contradiction. Therefore, Marty is telling the truth. Therefore, Marty did not eat the water crystal, and also not the air or fire crystal, so Marty ate the earth crystal.

Coyote Caoti says that Donna is telling the truth. Thus, if Donna is telling the truth, then Coyote Caoti is telling the truth, and if Donna is lying, then Coyote Caoti is lying.

Since Marty was already telling the truth, an only 2 apprentices are telling the truth, Coyote Caoti and Donna cannot both tell the truth, so they are both lying. Since only 2 apprentices are lying, Dr. Whoo is telling the truth. Since the consequent of Dr. Whoos statement is false, the antecedent of his statement must also be false. Thus, Donna did not eat the fire crystal, and since she lied, she ate the air crystal. That leaves the fire crystal for the lying Coyote Caoti and the water crystal for Dr. Whoo.

Grading rubric:

- 1pt for correctly explaining why Marty ate the earth crystal
- 1pt for the notion that Coyote Caoti and Donna are both telling the truth or both lying
- 1pt for a correct conclusion

End of the exam