Endterm Reasoning and Logic (CSE1300)

2021–22

Please read the following information carefully!

- This exam consists of 11 open questions. The open questions are worth a total of 57 points and the points per open (sub) question are given in the (sub) question itself.
- The grade for this exam is computed as $1 + 9 \cdot \frac{\text{score}}{57}$. Note that you require a grade of at least 5 out of 10 to pass the course (assuming your average course grade is at least 5.75).
- This exam corresponds to all non-starred sections of the book: *Delftse Foundations of Computation* (version 2.0).
- You have 180 minutes to complete this exam.
- Before you hand in your answers, check that every sheet of answer paper contains your name and student number.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Read every question carefully and, in the case of the open questions, give **all information** requested. Do not however give irrelevant information: this could lead to a deduction of points.
- You may write on this exam paper and take it home.
- No marmots or hippos were harmed in the creation of this exam.
- Exam prepared by S. Hugtenburg, S. Dumančić, I. van Kreveld, N. Yorke-Smith
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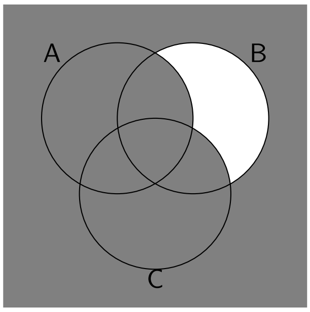
Question:	1	2	3	4	5	6	7	8	9	10	11	Total:
Points:	5	5	3	5	4	6	5	6	5	8	5	57

Learning goals coverage, based on the objectives of all lectures (strongly paraphrased):

Goal	mc 18	mt 18	et 18	rt 18	mc 19	mt 21	et 21	rt 21
translate logic to and from natural language	1,2	3,4	1	1	1,16-17	3-4	1, 10b	
describe $\wedge$,$\vee$,$\neg$,$\rightarrow$, and $\leftrightarrow$ operators					2			
construct a truth table	3-5	1a,1b	31a	21a	3-4	1a		
determine prop. logic equivalence	6,7,19		2		5-7	1b		
rewrite logical connectives	8-10		31b	21c	8	1c,d	2a	
describe contrapositives, converses, and inverses.	11,12			2	9		2b	
describe logic validity	13,14		3		10,11			
describe sufficient and necessary conditions	15		4		12	1c		
prove validty of argument in prop. logic	16-17	1b		3, 21b				
describe the principle of explosion	18	1c			13			
explain why prop. logic is not suf. expressive	20							
describe $\forall$ and $\exists$ quantifiers	21	2c	5					
evaluate negation stmt. in pred. logic	22			4	14-15		3a	
construct a Tarski's world	23-25							
construct a formal structure in pred. logic	26-27	2b	32a	22a	18	2a	3b	
evaluate claims about formal structures	28-29	2a	6,32b	22b	19	2b		
construct counterexamples for claims	30	2a,2b,5c		5	20	2c	3c	
describe the number sets $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$				6				
describe the form of a proof by div. into cases		5b	7			5a		
describe the form of a proof by contradiction				7				
construct a proof by division into cases		7a						
construct a proof by contradiction						5b	3a	
explain what a theorem prover is.			8					
describe the form of a proof by contrapositive		5a						
construct a proof by contrapositive		7b					4	
describe the form of a proof by generalisation.		5a,5b						
construct a proof by generalisation								
construct an existence proof			9					
identify type of proof to use for a given claim		5b				5b,8		
compute a sequence given a recursive definition		6a	10	8		7a		
construct and interpret recursive definitions		6b,6c						
explain the basic principle of an induction proof			11	9		6a		
construct an ind proof for numbers			33a	23a		6b	5a	
construct a tree based on a description						7b	6a	
construct an ind proof for algorithms			33b	10			10a	
construct recursive definitions on sets			12,13	23b, 24				
construct a proof using structural induction			14,15	23c			6b	
explain and apply basic set operations.			16	11				
construct a graph based on a description							7	
construct Venn diagrams			17,18	12			8a	
construct counterexamples for claims on sets			19,34b	25			8b	
compute the powerset of a set			20,21	13			8c	
compute the cartesian product of two sets			22	14				
construct proofs for claims on sets			34a	25				
describe Cantor's proofs about infinite sets			23	15			9a	
construct f or R from nat. language			24,25	26a			6c	
describe the diff. between f and R			35a	16				
determine the inverse of R and f			35b	26b				
determine if f is well-defined			26	17			9b	
determine if f is injective, surjective, or bijective			27	18			9a	
determine if R is symmetric, transitive or reflexive			28,29	19				
describe the properties of an equivalence relation			30	20, 26c			7c	

Open questions

1. (a) (1 point) Which set is shown in **grey**? Give your answer in terms of A, B, and/or C.



Answer: $(B - A - C)^c$, or $(B - (A \cup C))^c$, or one of many equivalent statements. **Grading rubric:**

- 1pt correct answer in Set Notation

(b) (2 points) If the following claim is true, prove it. If it is false, provide a counterexample and a clear explanation detailing how the counterexample shows the claim is false. Start your answer with either the word *True*, or *False*.

For all sets A, B, and non-empty sets C: $((A \cap B \neq \emptyset) \wedge (C \subset B)) \rightarrow A \cap C \neq \emptyset$

Answer: False, take $A = \{1, 2\}, B = \{1, 2, 3\}, C = \{3\}$. Now clearly the intersection between A and B is non-empty as it contains $\{1, 2\}$, C is a proper subset of B as it contains 3 but not all elements, yet the intersection between A and C is empty!

Grading rubric:

- 1pt correct counterexample
- 1pt correct explanation showing that the first half is true and the second half is false.

(c) (2 points) Compute $\mathscr{P}(A)$ given that $A = \{(0, 1), \emptyset, 2\}$.

Answer: $\mathscr{P}(A) = \{\emptyset, \{(0, 1)\}, \{2\}, \{\emptyset\}, \{\emptyset, (0, 1)\}, \{\emptyset, 2\}, \{(0, 1), 2\}, \{(0, 1), \emptyset, 2\}\}$

Grading rubric:

- 1pt at least 4 elements correct, and a total of 8 elements.
- 1pt fully correct answer

2. (5 points) Given a bag with 10 white beans, 15 grey beans and 20 black beans. As long as there are at least 2 different colours of beans in the bag, you follow the following procedure.

You take 2 beans of a different colour out of the bag, and you throw a bean of the other colour back in the bag. For example, when you would take a white and a black bean from the bag, you would throw a grey bean back in.

Once there is only a single colour of beans left in the bag, you cannot continue the procedure, as you cannot take 2 different colour beans from the bag.

If you try this procedure a couple of times, each time starting with 10 white beans, 15 grey beans and 20 black beans, you will notice that at the end of the procedure, there will always be the same colour of bean left! Which colour is it?

Prove this fact using an invariant. In your proof, use the following invariant: the parity of the amount of white beans is the same as the parity of the amount of black beans, but different from the parity of the amount of grey beans. (Note that parity means whether a number is odd or even.)

Answer:

Proof. We will prove this using an invariant.

Basis property: At the start, the amount of white beans is 10 and the amount of black beans is 20. These are both even numbers, so the parity of the amount of white beans and the amount of black beans is the same. The amount of grey beans is 15, which is odd, which means the parity of the amount of white beans and the amount of grey beans is different.

Inductive property: We assume the parity of the amount of white beans and the amount of black beans is the same, and the parity of the amount of white beans and the amount of grey beans is different. When we take 2 beans from a different colour from the bag, we throw one back in from the third colour. The amount of beans from 2 of the colours lowers by 1, while the amount of beans from the third colour raises by 1. This means that the parity of the amount of beans from a colour changes for all colours. Since the parity of the amount of white beans and the amount of black beans was the same and they both change, they stay the same. And since the parity of the amount of white beans and the amount of grey beans was different and they both change, they stay different.

Falsity of guard: Each step, the amount of total beans lowers by 1. For the sake of contradiction, let's assume the procedure will never terminate. In that case, since the amount of total beans lowers by 1 with each step, after an amount of steps equal to 1 lower than the total amount of beans, there is 1 bean left. This bean has a certain colour, which means that there are no beans of the other colours left, which means the procedure terminates. This contradicts our assumption. Thus, the procedure will terminate eventually.

Correctness of the post-condition: Since there are 2 colours of which there are no beans left, the amount of beans of those colours is 0. This means those colours cannot be white and grey, as those amounts had a different parity. Those colours also cannot be grey and black, as black has the same parity as white and thus a different parity to grey. Thus, the colours with 0 beans must be white and black, and the colour of the beans left must be grey. QED

Grading rubric:

- 1pt for a correct basic property
- 1pt for a correct inductive property
- 1pt for a correct argument of the falsity of the guard
- 1pt for the correct argumentation for the correctness of the post-condition
- 1pt for the correct answer.

3. Consider the following constants, predicates, and functions:

- The domain of discourse is 'all objects' (including living beings).
- $Marmot(x)$ means: x is a marmot
- A is the set of all animals
- a is for Amy, it is given that $a \in A$
- m is for Maximillion (or just Max for short), it is given that $m \in A$.
- $w : A \to \mathbb{N}$ is a function that gives the weight of an animal in kilograms.

Translate the following phrases to natural language (English).

(a) (1 point) $\exists B(B \in \mathscr{P}(A) \land \exists x(x \in B) \land \forall x \in B(w(x) > w(a)))$

Answer: Amy is not the heaviest animal. **Grading rubric:**

- 1pt Correct or not

(b) (2 points) $\exists E(\forall x(Marmot(x) \leftrightarrow x \in E) \land \forall x(x \in E \to w(x) < 12))$

Answer: All marmots weigh less than 12kg. **Grading rubric:**

- 1pt At least correctly recognise E is the set of all marmots.
- 1pt At least correctly recognise that everything in E weighs less than 12kg.

4. (a) (3 points) Rewrite $(p \to \neg(q \vee \neg r)) \to (p \wedge q)$ to CNF, simplify your result as much as possible. [Unsimplified answers can score at most 2 points.]

Answer:

$$\begin{aligned}
(p \to \neg(q \vee \neg r)) \to (p \wedge q) &\equiv \neg(p \to \neg(q \vee \neg r)) \vee (p \wedge q) \\
&\equiv (p \wedge (q \vee \neg r)) \vee (p \wedge q) \\
&\equiv ((p \wedge (q \vee \neg r)) \vee p) \wedge ((p \wedge (q \vee \neg r)) \vee q) \\
&\equiv (((p \vee p) \wedge (q \vee \neg r \vee p))) \wedge (((p \vee q) \wedge (q \vee \neg r \vee q))) \\
&\equiv p \wedge (q \vee \neg r \vee p) \wedge (p \vee q) \wedge (q \vee \neg r) \\
&\equiv p \wedge (q \vee \neg r)
\end{aligned}$$

Alternatively with a K-map:

p \ qr	00	01	11	10
0	0	0	0	0
1	1	0	1	1

This gets us $\neg(\neg p \vee (\neg q \wedge r)) \equiv p \wedge (q \vee \neg r)$

Grading rubric:

- 1pt for getting rid of both implications.
- 1pt for a final answer in CNF (even if not equivalent to the start)
- 1pt for a correct final answer simplified and in CNF.
- If a K-map is used: 1pt for the map, 1pt for the expression for the zeroes in DNF, 1pt for negating that to the correct CNF expression.

(b) (2 points) Consider the following claim:
"It is impossible for an implication and it's inverse to both be true at the same time."
Is this claim true or false? Explain your answer.

Answer: This claim is false, the inverse of an implication $p \to q$ is $\neg p \to \neg q \equiv q \to p$. So if a bi-implication holds, both the implication and its inverse hold. **Grading rubric:**

- 1pt Correctly tell us what the inverse is.
- 1pt Correct counterexample.

5. Consider the following definition of a recursive structure:

I. **End** $(x, \emptyset) \in A$ if $x \in \mathbb{N}$

II. **Sum** $(a_1, +, a_2) \in A$ if $a_1, a_2 \in A$

III. **Product** $(a_1, *, a_2) \in A$ if $a_1, a_2 \in A$

IV. **Exclusivity** Nothing else is in A

(a) (1 point) Give an example of an element $a \in A$ which you construct by applying rule I, II and III at least once each.

Answer: Starting with $(1, \emptyset)$ from rule 2, we can create $((1, \emptyset), +, (1, \emptyset))$ with rule 3 and then $((1, \emptyset), *, ((1, \emptyset), +, (1, \emptyset)))$ with rule 4. **Grading rubric:**

- 1pt for correct or not

(b) (3 points) Construct function $f : A \times \mathbb{N} \to A$ so that all odd values appearing in the structure are replaced with the second argument to f.

Answer: $f(a, y) = \begin{cases} (y, \emptyset) & \text{if } a = (x, \emptyset) \wedge 2 \nmid x \\ (x, \emptyset) & \text{if } a = (x, \emptyset) \wedge 2 \mid x \\ (f(a_1, y), +, f(a_2, y)) & \text{if } a = (a_1, +, a_2) \\ (f(a_1, y), *, f(a_2, y)) & \text{if } a = (a_1, *, a_2) \end{cases}$

Grading rubric:

- 1pt for at least having cases similar to the definition of A
- 1pt for correctly splitting the End rule into two options
- 1pt for correctly recursing in the other cases

6. (a) (1 point) Imagine you want to prove the following claim $A \subseteq B$ by using a proof by contradiction. What should be the starting assumption of your proof? Your answer should start with a quantifier.

Answer: $\exists x(x \in A \wedge x \notin B)$ **Grading rubric:**

- 1pt Correct or not

(b) (3 points) Prove the following set of statements is satisfiable, in other words give sets A, B, C, such that the following statements all hold. Briefly explain why all statements hold.

- $|A|, |B|, |C| \leq 3$
- $A \subseteq B$
- $(B \cap \mathscr{P}(C)) \neq \emptyset$
- $A \in \mathscr{P}(C)$

Answer: For example: $A = \{1\}, B = \{1, \emptyset\}, C = \{1, 2\}$. A is clearly a subset of B, $\mathscr{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, so this has the empty set in common with B and A is an element of it.
Grading rubric:

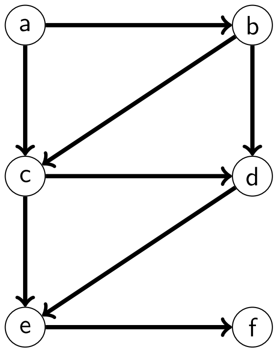
- 1pt for each statement satisfied, beyond the first one.
- -1pt of the total if a set has more than 3 elements.

(c) (2 points) Prove the following claim is false by providing a counterexample and a clear explanation as to why the counterexample shows the claim is invalid.
$\forall n \in \mathbb{N}(4 \mid n \vee \exists m \in \mathbb{Z}(m \mid n \wedge m > 1))$

Answer: The only counterexample is 1, as we can rewrite this claim to read: all numbers not divisible by 4 have a divisor larger than 1. This is true for all numbers (except 1), since they divide themselves. So you can always take $m = n$ to satisfy the consequent, except when $n = 1$ as in that case $m > 1$ will fail to hold!
Alternatively just explain that 1 doesn't work: 4 does not divide 1, nor is there a divisor of 1 larger than 1. **Grading rubric:**

- 1pt correct counterexample
- 1pt correct explanation that 1 is not divisible by 4, nor does it have a divisor larger than 1 (or alternative explanation with claim rewritten to implication)

7. (a) (2 points) Draw a DAG $G = (V, E)$ such that the topological ordering is: (a, b, c, d, e, f) and $8 \leq |E| \leq 12$.



Answer:

Grading rubric:

- 1pt DAG with 6 vertices, 8+ edges
- 1pt this is topological ordering

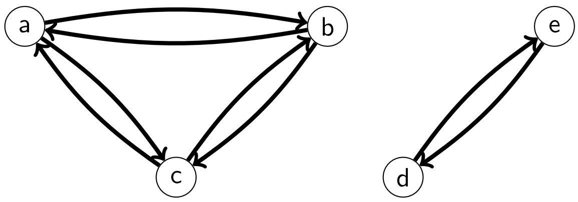
(b) (2 points) A connected component C in a graph is a set of vertices such that from any vertex $c \in C$ there is a simple path to all other vertices in C. What are the connected components of the following graph $G_1 = (V_1, E_1)$? List them as sets of vertices.

$V_1 = \{a, b, k, n, p, q, x, y, z\}$ $E_1 = \{\{x, n\}, \{a, k\}, \{k, z\}, \{p, x\}, \{z, y\}, \{y, a\}, \{n, b\}\}$

Answer: $\{x, n, p, b\}$, $\{a, k, z, y\}$, $\{q\}$ **Grading rubric:**

- 1pt at least one correct.
- 1pt all three correct.
- -1pt Not in set notation

(c) (1 point) For directed graphs we can consider E to be a relation on V^2. Now consider the following graph:



If E is an equivalence relation, give $[a]$. Otherwise indicate clearly why E is not an equivalence relation.

Answer: It is not reflexive, so not an equivalence relation! **Grading rubric:**

- 1pt correct or not

8. (6 points) Consider an adapted version of our TREE definition, that defines full binary trees (where every node has either 0 or 2 children).

1. **Leaf** $(x, \emptyset) \in FBTREE$ if $x \in D$
2. **Internal** $(x, (t_1, t_2)) \in FBTREE$ if $x \in D$ and $t_1, t_2 \in FBTREE$
3. **Exclusivity** Nothing else is in $FBTREE$

Prove that for all $t \in FBTREE$ the number of internal nodes is exactly one less than the number of leaf nodes.

You should use structural induction and should start by defining two functions to formalise the claim.

Answer:

Proof. Proof by structural induction. We define two functions $i : FBTREE \to \mathbb{N}$ and $l : FBTREE \to \mathbb{N}$ for the number of internal nodes and the number of leaf nodes in a tree respectively. The claim to be proven then reads: $\forall t \in FBTREE(i(t) + 1 = l(t))$

Base case ($t = (x, \emptyset)$ for some arbitrary $x \in D$): The number of leafs here is 1 and the number of internal nodes is 0. So indeed $l(t) = 1 = 0 + 1 = i(t) + 1$

Inductive step: Take some arbitrary $s_1, s_2 \in FBTREE$ so that $i(s_1)+1 = l(s_1)$ and $i(s_2)+1 = l(s_2)$. Now take an arbitrary $k \in D$ and construct $s_3 = (k, (s_1, s_2))$.

To prove: $i(s_3) + 1 = l(s_3)$

$l(s_3) = l(s_1) + l(s_2) \overset{\text{IH}}{=} i(s_1) + i(s_2) + 2 = (i(s_1) + i(s_2) + 1) + 1 = i(s_3) + 1$

(Explanation the number of internal nodes grows by one when using the internal node rule, but the number of leafs stays the same!)

Since s_1, s_2 were arbitrarily chosen, it holds for all trees that applying rule 3 maintains the property. Thus by structural induction we have shown that for all $t \in FBTREE$ $i(t) + 1 = l(t)$.

QED

Grading rubric:

- 1pt for introducing functions and formalising the claim
- 1pt for the base case
- 1pt for IH (taking two arbitrary objects)
- 1pt for correctly remarking we also need arbitrary $x \in D$ in the base case and/or inductive step.
- 1pt for the induction step
- 1pt for the conclusion

9. (5 points) Prove that $\forall n \in \mathbb{N}(3 \nmid 2n^2 + n + 1)$

Answer:

Proof. Take an arbitrary $k \in \mathbb{N}$. We exhaustively divide the proof into three cases:

- $k = 3m$ for some integer m: $2(3m)^2 + 3m + 1 = 18m^2 + 3m + 1 = 3(6m^2 + m) + 1$ which is clearly not divisible by 3.
- $k = 3m+1$ for some integer m: $2(3m+1)^2 + (3m+1) + 1 = 2(9m^2 + 6m + 1) + 3m + 1 + 1 = 18m^2 + 9m + 4 = 3(6m^2 + 3m + 1) + 1$, which is clearly not divisible by 3.
- $k = 3m+2$ for some integer m: $2(3m+2)^2 + (3m+2) + 1 = 2(9m^2 + 12m + 4) + 3m + 2 + 1 = 18m^2 + 27m + 11 = 3(6m^2 + 9m + 3) + 2$, which is clearly not divisible by 3.

Since the claims holds in all cases and k was arbitrarily chosen, it holds for all $n \in \mathbb{N}(3 \nmid 2n^2 + n + 1)$.

QED

Grading rubric:

- 1pt for arbitrary $k \in \mathbb{N}$.
- 1pt for correct 3 cases
- 1pt for correct math in at least one case
- 1pt for correct math in all cases
- 1pt for conclusion

Rubrics for other proof techniques added in grading platform.

10. (a) (2 points) Is the set $A = \{x\sqrt{2} \mid x \in \mathbb{Z}\}$ countably infinite? If so, give a bijection from $\mathbb{N}$ (there is no need to prove your function is bijection). If not, explain why not.

Answer: Yes it is! Sure the numbers themselves might be real numbers, but they are bound by $\mathbb{Z}$ in the definition, so a small modification of that bijection works!

$f : \mathbb{N} \to A$, with $f(n) = \begin{cases} n/2 \cdot \sqrt{2} & \text{if } 2 \mid n \\ -(n+1)/2 \cdot \sqrt{2} & \text{else} \end{cases}$ **Grading rubric:**

- 2pt for correct bijection
- 1pt for giving a bijection from $\mathbb{Z}$ to A instead.

(b) (2 points) Take the function $f : \mathbb{N} \to \mathbb{N}$ such that: $f(x) = \begin{cases} 1 & \text{if } x < 3 \\ 2f(x/2) - 3 & \text{if } 2 \mid x \\ 3f((x+1)/2) + 4 & \text{else} \end{cases}$

Is f well-defined? If so, explain why. If not, give a clear example of an input and computation of f on that input that show f is not well-defined.

Answer:

- $f(0) = f(1) = f(2) = 1$
- $f(3) = 3f(2) + 4 = 7$
- $f(4) = 2f(2) - 3 = 2 - 3 = -1$

$f(4) \notin \mathbb{N}$ and thus f is not well-defined! **Grading rubric:**

- 1pt for correct counter example
- 1pt for correct maths

(c) (2 points) A bijective function satisfies two properties. Give the names of both properties as well as a definition written in logic.

Answer: For a function $f : A \to B$:

- Injectivity: $\forall x, y \in A(f(x) = f(y) \to x = y)$
- Surjectivity: $\forall y \in B \exists x \in A(f(x) = y)$

Grading rubric:

- 1pt each, correct or not. Only minor spelling mistakes in name are forgiven, but if it means something else or the name is missing, it is wrong.

(d) (2 points) Now consider the function $g : \mathbb{N} \to \mathbb{Z}$ such that $g(x) = x^2 + 3$. Explain for each of the properties from the previous question whether they hold or not.

Answer:

- Injectivity: holds. A formal method involves showing that since the derivative is $2x$ the function is always increasing and so no two numbers will map to the same number. Less formal, but still sufficient for this course, is to explain that since we only have positive numbers the outcomes of x^2 will be unique and thus when adding three will remain unique.
- Surjectivity: does not hold. Take for example $g(x) = -10$ there is no input to make this works as $g(x) > 0$ for all x

Grading rubric:

- 1pt for each.

11. (5 points) Prove the following claim is true using mathematical induction:

For all integers $n \geq 2 : \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Answer: This exercise is taken from our book! :) So if it looked familiar, well done on preparing well!

Proof. Proof by mathematical induction. We define $P(n)$ to mean $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Base case (n = 2): $\sum_{i=0}^{2} 2^i = 2^0 + 2^1 + 2^2 = 7 = 8 - 1 = 2^3 - 1 = 2^{2+1} - 1$.

Induction step: Take some arbitrary $k \geq 2$ such that $P(k)$ holds (IH), we now prove that $P(k+1)$ holds as well.

$$\begin{aligned}\sum_{i=0}^{k+1} 2^i &= 2^{k+1} + \sum_{i=0}^{k} 2^i \\ \text{(by IH)} &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2^{k+2} - 1\end{aligned}$$

Since k was arbitrarily chosen it holds for all n that $P(n) \to P(n+1)$. Thus by the principle of induction we have shown that for all $n \geq 2 : \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. QED

Grading rubric:

- 1pt for the base case (also for $n = 0$ or $n = 1$ iff they explicitly mention that if it holds for all $n \geq 0/1$ it must also hold for all $n \geq 2$)
- 1pt for the IH, including notion of arbitrary
- 1pt for applying IH correctly
- 1pt for the rest of the math
- 1pt for the conclusion

End of the exam