# Endterm Reasoning and Logic (CSE1300)

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**Please read the following information carefully!**

- This exam consists of 20 multiple-choice questions and 6 open questions.
- The points for the multiple-choice part of the exam are computed as $1 + 9 \cdot \max(0, \frac{\text{score} - 0.25 * 20}{0.75 * 20})$. This accounts for a 25% guessing correction, corresponding to the four-choice questions we use.
- The grade for the open questions is computed as: $1 + 9 \cdot \frac{\text{score}}{66}$.
- The **final grade for the exam** is computed as: $0.4 \cdot MC + 0.6 \cdot Open$.
- This exam corresponds to all chapters of the book: *Delftse Foundations of Computation* (version 1.01).
- You have 3 hours to complete this exam.
- Before you hand in your answers, check that the sheet contains your name and student number, both in the human and computer-readable formats.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Note that the order of the letters next to the boxes on your multiple-choice sheet may **not always be A-B-C-D**!
- Tip: mark your answers on this exam **first**, and only after you are certain of your answers, copy them to the multiple-choice answer form.
- Read every question properly and in the case of the open questions, give **all information** requested, which should always include a brief explanation of your answer. Do not however give irrelevant information – this could lead to a deduction of points.
- Note that the minimum score per (sub)question is 0 points.
- You may write on this exam paper and take it home.
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Open questions:

| Question: | 21 | 22 | 23 | 24 | 25 | 26 | Total: |
|---|---|---|---|---|---|---|---|
| Points: | 11 | 8 | 11 | 11 | 13 | 12 | 66 |

# Multiple-Choice questions

1. (1 point) Which of the following claims **cannot** be **true** for two sets $A$ and $B$?
   A. $\mathscr{P}(A) \subseteq A$
   B. $A \subseteq \mathscr{P}(A)$
   C. $A \cup \{1\} \in \mathscr{P}(A \cup \{1\})$
   D. $A \cup B \in \mathscr{P}(A) \cup \mathscr{P}(B)$

2. (1 point) Which of the following translations of the statement 'Sean and Michelle solve several different crimes.' to predicate logic is **most** accurate? The predicates used are:
   - $s$ for Sean.
   - $m$ for Michelle.
   - $Crime(x)$ for $x$ is a crime.
   - $Solve(x, y)$ for $x$ solves $y$.

   A. $\exists x(Crime(x) \land (Solve(s, x) \lor Solve(m, x)))$
   B. $\forall x(Crime(x) \to (Solve(s, x) \to Solve(m, x)))$
   C. $\exists x, y(Crime(x) \land Crime(y) \land (x \neq y) \land (Solve(s, x) \land Solve(m, x)) \land (Solve(s, y) \land Solve(m, y)))$
   D. $\forall x(Crime(x) \land \exists y(Crime(y) \land (x \neq y) \land (Solve(s, x) \land Solve(m, x)) \land (Solve(s, y) \land Solve(m, y))))$

3. (1 point) Consider the following argument:
   $A \subseteq B$
   $(B \cap C) \neq \emptyset$
   $\therefore A \subseteq (B \cap C)$

   Which of the following presents a counterexample to this argument?
   A. $A = B = C = \{1, 2, 3\}$
   B. $A = B = \{1, 2, 3\}, C = \{1, 2\}$
   C. $A = C = \{1, 2, 3\}, B = \{1, 2\}$
   D. $B = C = \{1, 2, 3\}, A = \{1, 2\}$

4. (1 point) Which of the following functions is **not** well-defined?
   A. $L : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ with $L(n, m) = \dfrac{n + 4}{m - 3}$.
   B. $I : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ with $I(n, m) = n^{m+3}$.
   C. $N : \mathbb{N} \to \mathbb{Z}$, with $N(n) = \begin{cases} -n & \text{if } n \text{ is odd} \\ n + 1 & \text{else} \end{cases}$.
   D. $K : \mathbb{R} \to \mathbb{Q}$ with $K(n) = \dfrac{\lceil n \rceil}{2}$. Note that $\lceil x \rceil$ is to round $x$ up to the closest integer.

5. (1 point) Consider the following templates for a proof by induction. Which one does **not** prove their claim over natural numbers? Note: The claims also differ from answer to answer.
   A. Claim: $\forall n \geq 0(P(n))$.
   Base case: prove $P(0) \land P(1)$.
   Inductive step: prove $P(k) \to P(k + 2)$ holds for all $k \geq 0$.
   B. Claim: $\forall n \geq 5(P(n))$.
   Base case: prove $P(0) \land P(3)$.
   Inductive step: prove $P(k) \to P(k + 1)$ holds for all $k \geq 5$.
   C. Claim: $\forall n \geq 5(P(n))$.
   Base case: prove $P(5)$.
   Inductive step: prove $\neg P(k + 1) \to \neg P(k)$ holds for all $k \geq 5$.
   D. Claim: $\forall n \geq 0(P(n))$.
   Base case: prove $P(0)$.
   Inductive step: prove $P(k + 1) \to P(k + 2)$ holds for all $k \geq -1$.

6. (1 point) Which of the following **cannot** be used to show that two propositions $P$ and $Q$ are equivalent?
   A. Show that $(P \to Q) \wedge (Q \to P)$ is a tautology.
   B. Show that $\neg Q \to \neg P$ and $\neg P$ are both contradictions.
   C. Show that $(P \to R) \wedge (R \to Q) \wedge (Q \to P)$ is a tautology.
   D. Show that $P$ is a contradiction and that $\neg Q$ is a tautology.

7. (1 point) Consider the following informal description of a relation $R$. $R$ is a binary relation from the set $A = \{0, 1, 2\}$ to the set $B = \{a, b, c\}$, in such a way that every element from $A$ has a relation to exactly 2 unique items in $B$.

   Which of the following describes $R$ in set-notation?
   A. $R = A \times B$
   B. $R = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}$
   C. $R = \{(0, 1), (1, 2), (2, 0), (a, a), (b, b), (c, c)\}$
   D. $R = \{(a, 0), (b, 0), (a, 2), (b, 1), (c, 2), (c, 1)\}$

8. (1 point) Consider the argument: $\begin{array}{c} A \\ B \\ \hline \therefore C \end{array}$. Which of the following methods can we always use to prove the validity of the argument?
   A. Show that $\frac{A}{\therefore C}$ is a valid argument.
   B. Show that $\neg(A \to B) \to \neg C$ is a tautology.
   C. Use equivalence rules to derive that $A \equiv B$ and that $B$ is a tautology.
   D. Construct a truth table for $A$, $B$, and $C$ and inspect all rows in which $C$ is false. The argument is valid if and only if both $A$ and $B$ are also false in those rows.

9. (1 point) Which of the following claims about sets $A, B, C$ is always **true**?
   A. $|(A \cap B) \cup C| \geq |A \cap (B \cup C)|$
   B. $|A \cup B| > |A|$ and/or $|A \cup B| > |B|$
   C. $|A \cap B| \geq |A \cap C|$ if ($|B| \geq |C|$ and $|A| > 0$)
   D. $|A \cap B| > 0$ if and only if ($|A| > 0$ and $|B| > 0$)

10. (1 point) In his TED talk "What do top students do differently?"[1] Douglas Barton claims that: "In our research we found that hard work was a necessary condition, but it wasn't a sufficient condition to doing well."

    Taking $p$ to mean 'working hard' and $q$ to mean 'doing well', which of the following is true?
    A. $p \leftrightarrow q$
    B. $p \to q \wedge \neg(q \to p)$
    C. $\neg(p \to q) \wedge (\neg p \to \neg q)$
    D. The claim is a contradiction, and can thus be expressed as $\mathbb{F}$.

11. (1 point) We want to prove the following claim over all natural numbers $\mathbb{N}$: $\forall n(4 \mid f(n) \vee 3 \mid f(n))$, where $f(n)$ is some function $f : \mathbb{N} \to \mathbb{N}$. What proof-outline sketched below is **not** suitable?
    A. Take an arbitrary $k$, show that $12 \mid f(k)$.
    B. Take an arbitrary $k$, show that $3 \nmid f(k) \to 4 \mid f(k)$ holds.
    C. Take an arbitrary $k$, show that $2 \mid k \to 4 \mid f(k)$ holds and that $3 \mid k \to 3 \mid f(k)$ holds.
    D. Take an arbitrary $k$, show that when $k = 2m$ it holds that $3 \mid f(k)$, and that when $k = 2m + 1$ it holds that $4 \mid f(k)$.

[1] https://www.youtube.com/watch?v=Na8m4GPqA30

12. (1 point) Consider the following formal structure $S$ with domain $\{p, a, m, f, k, g\}$:

- $L^S = \{p, a, g\}$
- $P^S = \{m, f, k\}$
- $W^S = \{(p, m), (p, f), (m, p), (a, p), (a, k), (k, p)\}$
- $B^S = \{(p, p), (m, m), (a, p), (g, g), (k, m)\}$

Which of the following claims is **true**?

A. $\exists x(L(x) \wedge \forall y(W(x, y)))$
B. $\forall x(L(x) \to \exists y(B(x, y) \wedge (x \neq y))$
C. $\forall x((L(x) \to P(x)) \vee (P(x) \to L(x)))$
D. $\exists x(P(x) \wedge \forall y((P(x) \wedge (x \neq y)) \to B(x, y)))$

13. (1 point) Consider the following recursive definition of a set of strings ('words') $S$:

I. $a \in S$.
II. if $x \in S$, then $xi \in S$, $axa \in S$.
III. if $x, y \in S$, then $ixiyixi \in S$.
IV. Nothing else is in $S$ other than the strings constructed with these rules.

Note that in this definition $a$ and $i$ are letters of the word and that $x$ and $y$ are variables, they represent arbitrary words in the set.

Which of the following claims over this set is **false**?

A. All words in the set have at least one $a$ in it.
B. There is a word in the set without an $i$ in it.
C. There is a word in the set with exactly five $a$'s in it.
D. All words in the set have an even numbers of $i$'s in it.

14. (1 point) Consider again the recursive definition from the previous Question 13. Which of the following rule(s) should we add to ensure that string $mia$ is a part of the language?

A. V. if $x \in S$, then $mx \in S$.
B. V. $i \in S$. VI. if $x \in S$, then $xm \in S$
C. V. $m \in S$. VI. if $x, y \in S$, then $xiy \in S$.
D. V. $m \in S$. VI. if $x, y \in S$, then $ixy \in S$.

15. (1 point) Figure 1 shows a Venn Diagram. Which of the following sets is represented by the shaded area?

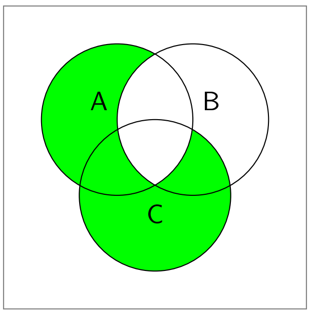
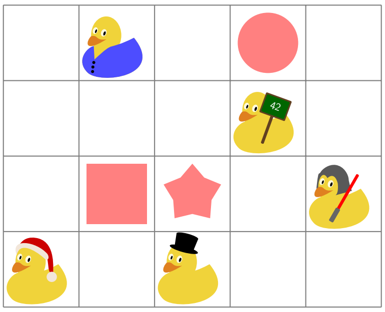



Figure 1: Venn Diagram

A. $(A \Delta B) \cup (C \Delta B)$
B. $(A \setminus B) \cup (C \setminus A)$
C. $(A^c \setminus B) \cup (C \setminus (A \cap B))$
D. $((C \setminus A) \cap (C \setminus B)) \cup (A \Delta B)$

16. (1 point) Consider now the following alternative visualisation of sets: a Duck Empire. The set $P$ contains all ducks that are within a distance of 2 from the circle, $Q$ contains all ducks within a distance of 2 from the star, and $R$ contains all ducks within a distance of 2 from the square.[2]




Which of the following claims is **true**?

A. $|P \cap Q| = 5$

B. $|Q \cup R| = 5$

C. $|R \times P| = 5$

D. $|P| + |Q| + |R| = 5$

17. (1 point) Which of the following statements is **true**?

A. $\mathscr{P}(\emptyset) = \emptyset$

B. $\mathscr{P}(\emptyset) = \emptyset \cup \{\emptyset\}$

C. $\mathscr{P}(\mathscr{P}(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

D. $\mathscr{P}(\mathscr{P}(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\}$

18. (1 point) Which of the following claims is **false**?

A. $\mathbb{N}$ has the same cardinality as $\{\pi^k \mid k \in \mathbb{Q}\}$.

B. $\mathbb{N}$ has the same cardinality as $\{x \mid x = 42k \text{ for some } k \in \mathbb{Z}\}$.

C. $\mathbb{N}$ has the same cardinality as $\{x \mid 0.1 \leq x \leq 0.11 \text{ for } x \in \mathbb{R}\}$.

D. $\mathbb{N}$ has the same cardinality as $\{x \mid x = 2k + 1 \text{ for some } k \in \mathbb{N}\}$.

[2] Ducks cannot travel diagonally, so the duck with the Santa hat is exactly 2 squares always from the square, but a distance 3 away from the star.

19. (1 point) Which of the following claims is **true** ?
    A. A relation that is not reflexive, cannot be transitive.
    B. A relation that is not transitive, cannot be symmetric.
    C. A relation that is both symmetric and transitive, has to be reflexive.
    D. All 8 permutations of relations that are (not) symmetric, (not) transitive and/or (not) reflexive can be created.

20. (1 point) As Lyra prepares to travel to the north, she draws up a map of the kingdom of the Panserbjørn (see Figure 2). On it she also draws the different roads available between the cities of the kingdom. Lyra's friend Pantalaimon realises this could also be a visualisation of a relation $P(x, y)$. Which of the following statements about this relation is true?

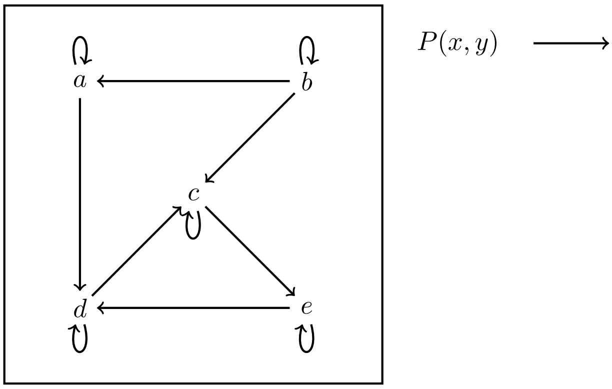



Figure 2: The map as drawn by Lyra

    A. $P$ is not currently transitive, but adding $P(a, c)$ and $P(b, e)$ would make it transitive.
    B. $P$ is currently reflexive, but removing $P(b, c)$ and adding $P(c, b)$ would make it lose its reflexivity.
    C. $P$ is currently transitive, but removing $P(a, d)$ and $P(e, d)$ and adding $P(a, c)$ and $P(c, d)$ from/to the relation would make it lose this property.
    D. $P$ is not currently symmetric, but removing $P(b, c), P(b, a), P(a, d), P(d, c)$ and adding $P(e, c)$, $P(c, a)$,$P(a, c)$, and $P(d, e)$ to the relation would make it symmetric.

# Open questions

21. Consider the following two statements written in propositional logic.

    I. $\neg((p \rightarrow r) \leftrightarrow q) \vee (p \rightarrow q)$

    II. $(p \wedge q) \vee \neg(q \rightarrow p)$

    (a) (6 points) Create truth tables for both statements. Clearly indicate with column(s) contain your final answer(s) and please order your rows starting with all zeroes for the propositions.

    (b) (1 point) Are the statements equivalent? Explain your answer in at most 3 lines.

    (c) (4 points) Rewrite statement II to CNF, simplifying the result as much as possible.

22. Consider the following properties of the integers $0$ up to and including $10$. Such a number has the property $E$ if it is even. It has the property $P$ if it is prime. It has the property $S$ if it is a square of an integer. Finally, two such numbers $x$ and $y$ have relation $R$ if $x \cdot y > 72$.

    (a) (6 points) Describe the structure above formally. Give the domain as well as the truth sets for the predicates corresponding to the properties and relation above in set-notation.

    (b) (2 points) Consider now the following extra information for this structure:
    $\forall x((P(x) \wedge E(x)) \rightarrow Z(x))$
    May we now conclude that $\exists x(Z(x))$ for this structure? Explain your answer in at most 5 lines.

23. (a) (6 points) Prove the following claim $\forall n \geq 1, n \in \mathbb{N} : \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.

    (b) Consider the following algorithm:

    **function** F(n)
        $x \leftarrow 0$
        $c \leftarrow 0$
        **while** $c \leq n$ **do**
            $c \leftarrow c + 1$
            $x \leftarrow x + c$
        **end while**
        **return** $x$
    **end function**

    Consider the invariant $x = \sum_{i=0}^{c} i$.

    i. (1 point) Prove that the invariant holds before the loop starts.

    ii. (3 points) Prove that the invariant holds after an iteration, assuming it holds before the loop.

    iii. (1 point) Explain why the algorithm terminates. Answer in at most 3 lines.

24. (11 points) Below are two claims over sets. One of the claims is true and a proof for that claim can earn you 6 points. One of the claims is false and a counterexample including clear explanation for that claim can earn you 4 points. Correctly indicating which one is true and which one is false gets you an additional 1 point, for a total of 11 points for this question. Indicate clearly which claim you believe to be true and which one false.

    (a) For three sets $A, B, C$: $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$

    (b) For three sets $A, B, C$: $(A \subset B \wedge B \cap C \neq \emptyset) \rightarrow A \cap C \neq \emptyset$

25. (a) (5 points) Give a recursive definition of the set $A$ that only contains the number $120$, and for any numbers in the set, the set also contains:

    - all of the factors/divisors of those numbers.
    - all of the products of those numbers.

    (b) (8 points) Consider again the recursive definition from Question 13. Prove the following claim: Each word in $S$ contains an odd number of $a$'s.

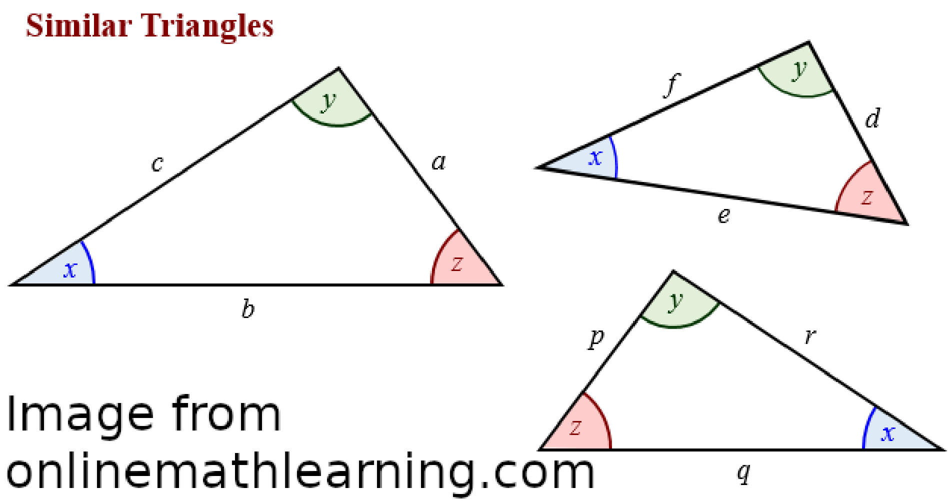



Figure 3: Examples of similar triangles.

26. (a) (2 points) Describe in your own words the two differences between functions and relations. Answer in at most 5 lines.

    (b) For each of the following functions/relations indicate if they have an inverse. If so, give it. If not, explain why not.

    i. (1 point) The function/relation $f : \mathbb{Q} \to \mathbb{Q}$, where $f(x) = 18x + 26$.

    ii. (1 point) The function/relation $g : \mathbb{Z} \to \mathbb{R}$, where $g(x) = x^2 + 5x - 12$.

    iii. (1 point) The function/relation $h = \{(a, b), (c, d), (e, f), (g, h), (i, j)\}$.

    iv. (1 point) The function/relation $\ell = \{(1, 1), (2, 2), (3, 3)\}$.

    (c) (6 points) Out of the following 2 relations, one is an equivalence relation and the other is not. For the relation that is not an equivalence relation, give a counterexample and explanation as to why it is not in at most 5 lines for 2 points. For the relation that is an equivalence relation, explain why it is (discuss *all* properties an equivalence relation should have) in at most 8 lines for 3 points. Correctly indicating which one is false and which one is true gets you 1 point, for a total of 6 points.

    i. A relation $R$ over triangles. We say two triangles are *similar* when they have the same set of angles. See Figure 3 for some examples of similar triangles. Let $(x, y) \in R$ for some triangles $x, y$ if and only if $x$ and $y$ are similar.

    ii. A relation $B$ over people. Let a person $x$ have a age $a_x$ and an hourly wage $w_x$. Then let $(x, y) \in B$ for some people $x$ and $y$ if and only if $w_x + w_y \leq a_x + a_y$.

End of the exam