Endterm Reasoning and Logic (CSE1300)

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Please read the following information carefully!

- This exam consists of 20 multiple-choice questions and 6 open questions.
- The points for the multiple-choice part of the exam are computed as $1 + 9 \cdot \max(0, \frac{\text{score} - 0.25 * 20}{0.75 * 20})$. This accounts for a 25% guessing correction, corresponding to the four-choice questions we use.
- The grade for the open questions is computed as: $1 + 9 \cdot \frac{\text{score}}{64}$.
- The **final grade for the exam** is computed as: $0.4 \cdot MC + 0.6 \cdot Open$.
- This exam corresponds to all chapters of the book: *Delftse Foundations of Computation* (version 1.01).
- You have 3 hours to complete this exam.
- Before you hand in your answers, check that the sheet contains your name and student number, both in the human and computer-readable formats.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Note that the order of the letters next to the boxes on your multiple-choice sheet may **not always be A-B-C-D**!
- Tip: mark your answers on this exam **first**, and only after you are certain of your answers, copy them to the multiple-choice answer form.
- Read every question properly and in the case of the open questions, give **all information** requested, which should always include a brief explanation of your answer. Do not however give irrelevant information – this could lead to a deduction of points.
- Note that the minimum score per (sub)question is 0 points.
- You may write on this exam paper and take it home.
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Open questions:

Question:	21	22	23	24	25	26	Total:
Points:	10	12	8	11	5	18	64

Multiple-Choice questions

1. (1 point) Consider the following statement:
   'Atticus is the only lawyer representing Tom.'

   Which of the following translations to predicate logic is **most** accurate? The predicates used are:

   - a for Atticus.
   - t for Tom.
   - $Lawyer(x)$ for x is a lawyer.
   - $Represents(x, y)$ for x represents y.

   A. $Lawyer(a) \land Represents(a, t) \land \neg\exists x(Lawyer(x))$

   B. $Represents(a, t) \land \forall x((Lawyer(x) \land (x \neq a)) \to \neg Represents(x, t))$

   C. $Lawyer(a) \land Represents(a, t) \land \exists x(Represents(x, t) \land (Lawyer(x) \land (x \neq a)))$

   D. $Lawyer(a) \land Represents(a, t) \land \forall x(Represents(x, t) \to (\neg Lawyer(x) \lor (x = a)))$

2. (1 point) Which of the following is **true**?

   A. If $p \to q$ is a tautology, then the converse is guaranteed to be a contradiction.

   B. If $p \to q$ is a contradiction, then the converse is guaranteed to be a tautology.

   C. If $p \to q$ is a tautology, then the inverse is guaranteed to be a contradiction.

   D. If $p \to q$ is a contingency, then the inverse is guaranteed to be a contingency.

3. (1 point) Consider the argument: $A, B \therefore C$. Which of the following methods can we use to prove the validity of the argument?

   A. Show that $B \land C$ is a contradiction.

   B. Show that $\neg A \land C$ is a contradiction.

   C. Show that $B \land \neg C$ is a contradiction.

   D. Show that $\neg A \land \neg B$ is a contradiction.

4. (1 point) Which of the following formulas is equivalent to: $\neg\forall x\exists y(P(x) \land (Q(y) \to \exists z(R(x, y, z))))$?
   Note: Carefully read the order of the quantifiers and their associated letters.

   A. $\exists y\forall x\neg(P(x) \land (Q(y) \to \exists z(R(x, y, z))))$

   B. $\exists x\forall y(\neg P(x) \lor (Q(y) \land \forall z(\neg R(x, y, z))))$

   C. $\exists y\forall x(\neg P(x) \lor (\exists z(\neg R(x, y, z)) \land Q(y)))$

   D. $\exists x\forall y(\neg P(x) \lor (\neg Q(y) \to \forall z(\neg R(x, y, z))))$

5. (1 point) Consider the following claims about the positive integers $\mathbb{N}$, where the predicate $P(x)$ is used to indicate that x is prime. Remember that $n \mid m$ is used to represent that n divides m. Which of these claims is true?

   A. $\forall x(((x < 10) \land (2 \nmid x)) \to P(x))$

   B. $\forall x((\exists y\exists k((x = yk))) \to \neg P(x))$

   C. $\neg\forall x(\neg P(x) \lor \forall y((y \leq x) \lor (y \nmid x)))$

   D. $\neg\forall x(\neg P(x) \lor \neg\exists y(((2 < y) \land (y < x)) \land \neg P(y)))$

6. (1 point) Which of the following statements is **false**?

   A. $\sqrt{16} \in \mathbb{Z}$

   B. $\frac{\pi}{42\pi} \in \mathbb{Q}$

   C. $\log_2(3) \in \mathbb{Q}$

   D. $(\sqrt{-8})^2 \in \mathbb{R}$

7. (1 point) Consider the following proof:

   Proof. Assume $(\neg a \vee \neg b)$ and c to hold. <Derive contradiction> Therefore ... QED

   What may we conclude at the place of the ...?

   A. $(a \wedge b) \rightarrow c$

   B. $c \rightarrow (a \wedge b)$

   C. $(a \wedge b) \rightarrow \neg c$

   D. $\neg c \rightarrow (a \wedge b)$

8. (1 point) Consider the following recursively defined sequence s: $s_0 = 1$, $s_1 = 2$, $s_n = (s_{n-1} + n - 1) \cdot s_{n-2}$ for $n \geq 2$.

   What is the value of s_4?

   A. 5

   B. 21

   C. 39

   D. 57

9. (1 point) Consider a situation where we want to apply induction to prove a property for all integers in $\mathbb{Z}$. What do we now need to do in addition to the regular steps in mathematical induction?

   A. No additional steps are required.

   B. In the base case also show that it holds for $n = -1$.

   C. In the inductive step also show that $\forall n \leq 0(P(n) \rightarrow P(n+1))$ holds.

   D. In the inductive step also show that $\forall n \in \mathbb{Z}(P(n) \rightarrow P(n-1))$ holds.

10. (1 point) When proving a property of an algorithm with a simple while-loop using induction, we pick a certain invariant. Which of these do we **not** need to show?

    A. Show that the loop runs at least once.

    B. Show that the invariant holds before the loop.

    C. Show that the invariant holds at the end of the code.

    D. Show that if the invariant holds before the loop, it also holds after the end of the loop.

11. (1 point) Consider the following statements about three arbitrary non-empty finite sets A, B, and C. Which of the following is **guaranteed** to be **true**?

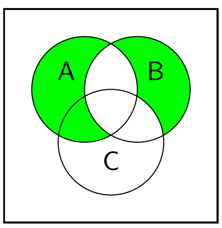
    A. If $A \subseteq B - C$, then $A \subseteq B \vee A \subseteq C$.

    B. If $A \subseteq B \Delta C$, then $A \subseteq B \vee A \subseteq C$.

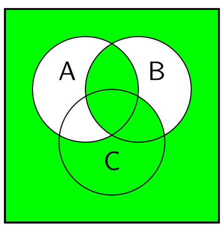
    C. If $A \subseteq B \cup C^c$, then $A \subseteq B \vee A \subseteq C$.

    D. If $A \subseteq B^c \cap C^c$, then $A \subseteq B \vee A \subseteq C$.

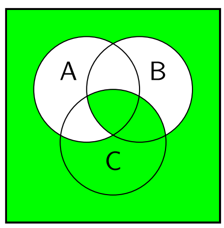
12. (1 point) Let A, B, C be finite non-empty sets. Take $D = ((A - B)\Delta(B - C))^c$. Which of the following diagrams represents this set?



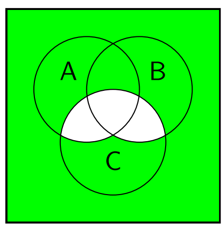
Answer A



Answer B



Answer C



Answer D

13. (1 point) Let A, B be arbitrary sets. What can we say with certainty?
    - A. If $A \in \mathscr{P}(B)$, then $A \in B$.
    - B. If $A \subseteq B$, then $A \in \mathscr{P}(B)$.
    - C. If $A = \mathscr{P}(B)$, then $A \subseteq B$.
    - D. If $A \in B$, then $\{A\} \subseteq \mathscr{P}(B)$.

14. (1 point) Consider the sets: $A = \{$Lyra,Jack,Lenny$\}$ and $B = \{$Jethro,Pan,Will$\}$. Which of the following is an element of $A \times B$?
    - A. $\{$Will,Lyra$\}$
    - B. (Lyra,Jethro)
    - C. $\{$(Will, Jack)$\}$
    - D. $(\{$Lenny$\}, \{$Pan$\})$

15. (1 point) Which of the following claims is **true**?
    - A. $\mathbb{N}$ has a strictly smaller cardinality than $\{\pi^k \mid k \in \mathbb{Q}\}$.
    - B. $\mathbb{N}$ has the same cardinality as $\{x \in \mathbb{R} \mid x = 2k$ for some $k \in \mathbb{N}\}$.
    - C. $\mathbb{N}$ has a strictly bigger cardinality than $\{x \mid x = 42k$ for some $k \in \mathbb{N}\}$.
    - D. $\mathbb{Q}$ has a strictly bigger cardinality than $\{x \mid x = 2k + 1$ for some $k \in \mathbb{N}\}$.

16. (1 point) Consider the following description of a function f. f takes a function g and an integer and returns a fraction. g is a function that takes a real number and an integer and returns a fraction and an integer. Which of the following describes the function f formally?
    - A. $f : (\mathbb{Q} \times \mathbb{N})^{\mathbb{R} \times \mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$.
    - B. $f : (\mathbb{R} \times \mathbb{N})^{\mathbb{Q} \times \mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$.
    - C. $f : (\mathbb{Q})^{\mathbb{R} \times \mathbb{N}} \times \mathbb{N} \to \mathbb{Q} \times \mathbb{N}$.
    - D. $f : (\mathbb{Q})^{\mathbb{Q} \times \mathbb{N}} \times \mathbb{N} \to \mathbb{R} \times \mathbb{N}$.

17. (1 point) Which of the following is **true**? *Note:* for this question you should assume that every x and y actually read: $x \in X$ and $y \in Y$.
    - A. In a relation $R \subseteq X \times Y$ there can be only one x for every y.
    - B. In a relation $R \subseteq X \times Y$ there can be only one y for every x.
    - C. In a well-defined function $f : X \to Y$, there can be only one x for every y.
    - D. In a well-defined function $f : X \to Y$, there can be only one y for every x.

18. (1 point) Consider the following function $L : \mathbb{N} \to \mathbb{Z}$, with $L(n) = \begin{cases} -\dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{else} \end{cases}$.

    Which of the following statements is **true**?
    - A. L is only surjective and not injective.
    - B. L is only injective and not surjective.
    - C. L has an inverse.
    - D. L is not well-defined.

19. (1 point) Now that Lyra has arrived in the kingdom of the Panserbjørn[1], she discusses the organisation of the bears with the king. The king indicates that for the sake of the kingdom it would be good to define a relation $C \subseteq B \times B$ where B denotes the set of all bears. We use the relation to express that a certain bear a would like to work with a certain bear b. The king says that ideally the relation is both symmetric and transitive. In that case, which of the following should be **true**?

    A. For every pair of bears: if bear x cannot cooperate with bear y, then y can cooperate with bear x.

    B. Every bear should be able to cooperate with a specific bear x and bear x should be able to cooperate with everyone.

    C. For every bear x that can cooperate with a bear y there should be a bear z such that both x and y can cooperate with z.

    D. For every bear x that cannot cooperate with a bear y, there is no bear z such that x can cooperate with z and z can cooperate with y.

20. (1 point) Lyra argues that although the bear king is wise and smart, he should instead aim to find an equivalence relation to divide the bears into groups of bears that can cooperate. After some time Lyra manages to find such a relation and partitions the bears into hunting teams. Which of the following is now **impossible**:

    A. There is only a single hunting team.

    B. Every hunting team is of the same size.

    C. There is a bear that is not part of any team.

    D. There is a hunting team comprised of only one bear.

[1]See endterm Reasoning & Logic 2018

Open questions

21. Consider the following two statements written in propositional logic.

    I. $(\neg p \vee \neg q) \rightarrow p$

    II. $r \rightarrow (p \vee (q \wedge (r \rightarrow \neg q)))$

    (a) (6 points) Create truth tables for both statements. Clearly indicate the column(s) that contain your final answer.

    (b) (1 point) Consider now the argument: $\frac{\text{I}}{\therefore \text{II}}$ where I and II refer to the full statements above. Is this argument valid? If so, explain why. If not, indicate clearly how your truth table shows this (give a concrete counterexample!)

    (c) (3 points) Rewrite statement II to a form that uses only $\neg$ and $\rightarrow$.

22. (a) For each of the following descriptions of a function, give a well-defined function that matches the description.

    i. (1 point) A function $f : \mathbb{Z} \rightarrow \mathbb{N}$

    ii. (2 points) A function $g : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Q}$

    iii. (1 point) A function $h : \mathbb{N} \rightarrow \{x \mid x = 7k \text{ for some } k \in \mathbb{N}\}$ that is bijective.

    (b) For each of the following *functions/relations*, give their inverse if they have one. If they do not have an inverse, give a clear and concrete example that shows *why* the function/relation does not have an inverse.

    i. (1 point) $f : \mathbb{Q} \rightarrow \mathbb{Z}$ where $f(x) = -6x + 17$.

    ii. (1 point) $g : \mathbb{N} \rightarrow \mathbb{N}$ where $g(x) = x^2$.

    iii. (1 point) $h : \{1, 2, 3\} \rightarrow \mathbb{N}$ where $h = \{(1, 2), (2, 3), (3, 2)\}$

    iv. (1 point) $l \subseteq \mathbb{N} \times \mathbb{N}$, with $l = \{(1, 2), (2, 3), (3, 2)\}$

    (c) (4 points) Out of the following 2 relations, one is an equivalence relation and the other is not. For the relation that is not an equivalence relation, give a counterexample and explanation as to why it is not in at most 5 lines for 1 point. For the relation that is an equivalence relation, explain why it is (discuss *all* properties an equivalence relation should have) in at most 8 lines for 2 points. Correctly indicating which one is an equivalence relation and which one is is not gets you 1 point, for a total of 4 points.

    i. The relation P between snails defined as follows: A snail a and a snail b are in the relation P iff they share at least one parent.

    ii. The relation T between snails defined as follows: A snail a and a snail b are in the relation T iff they share a terrarium (a housing unit for snails).

23. Construct **recursive** definitions for the two sets S and T below:

    (a) (4 points) The set $S \subseteq \mathbb{N}$ contains the numbers 6 and 7. Furthermore any sum or product of two numbers in S is also in S. Finally any number that is 5 away from any number in S on a number line is also in S. (For example both 2 and 12 are in S as they are 5 away from 7.)

    (b) (4 points) The set T contains the fraction $\frac{1}{2}$. Furthermore every number t in the set can be written as: $t = \frac{2^k}{2^l}$ where $k, l \in \mathbb{N}$.

24. (11 points) Below are two claims over sets. One of the claims is true and a proof for that claim can earn you 6 points. One of the claims is false and a counterexample including clear explanation for that claim can earn you 4 points. Correctly indicating which one is true and which one is false gets you an additional 1 point, for a total of 11 points for this question. Indicate clearly which claim you believe to be true and which one false.

    (a) For three sets A, B, C: $(A \subseteq B \wedge (B \cup C) \subseteq A) \rightarrow A = B$

    (b) For three **non-empty** sets A, B, C within some universe U: $(B - C)^c \cap (A \cup B) \subseteq A$

25. Consider the following set of words W: {snail, alexander, anay, phoenix, nick, maya, pearl, gerrit}. We define the following predicates over W. A is the set of all words that contain the letter a. B is the set of all words contain the same letter more than once. All pairs of words of length 4 and a word of length 8 that start with the same letter form the relation R.

   (a) (3 points) Describe the structure above formally. Give the predicates corresponding to the properties and relation above in set-notation.

   (b) (1 point) Consider now the following extra information for this structure:
   $\forall x((A(x) \wedge B(x)) \rightarrow Z(x))$
   A student claims that we can read the predicate Z in our structure as: Z is the set of all words that contain at least two a's. Is the student correct? If so, explain why. If not, give a counterexample.

   (c) (1 point) Consider now that we add the word *harry* to the set W. Is the translation of Z for this new set W correct? If so, explain why. If not, give a counterexample.

26. (a) (8 points) Consider the sequence: $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for all $n > 2$. Prove that $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ for all $n \geq 1$. *Hint:* use strong induction for your proof.

   (b) (2 points) Consider the following recursively defined set S:

   I. $\pi \in S$
   II. $\forall x : x \in S \rightarrow \pi + x \in S$
   III. $\forall x, y : x, y \in S \rightarrow \dfrac{x}{y} \in S$
   IV. Nothing else is in S.

   Explain why $\{x \in \mathbb{N} \mid x > 0\} \subseteq S$. Be clear what rules should be applied to form the elements from $\{x \in \mathbb{N} \mid x > 0\}$.

   (c) (8 points) Now consider the following recursively defined set T:

   I. $8 \in T$
   II. $\forall x : x \in T \rightarrow 7x \in T$
   III. $\forall x : x \in T \rightarrow 3x^2 + 2x + 18 \in T$
   IV. $\forall x, y : x, y \in T \rightarrow x^y \in T$
   V. Nothing else is in T.

   Prove that every element in T is even.

End of the exam