Endterm Reasoning and Logic (CSE1300)

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Please read the following information carefully!

- This exam consists of 16 multiple-choice questions and 9 open questions.
- The points for the multiple-choice part of the exam are computed as $1 + 9 \cdot \max(0, \frac{\text{score} - 0.25 * 0}{0.75 * 0})$. This accounts for a 25% guessing correction, corresponding to the four-choice questions we use.
- The grade for the open questions is computed as: $1 + 9 \cdot \frac{\text{score}}{65}$.
- The **final grade for the exam** is computed as: $0.4 \cdot MC + 0.6 \cdot Open$.
- This exam corresponds to all non-starred sections of the book: *Delftse Foundations of Computation* (version 1.1).
- You have 3 hours to complete this exam.
- Before you hand in your answers, check that the sheet contains your name and student number, both in the human and computer-readable formats.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Note that the order of the letters next to the boxes on your multiple-choice sheet may **not always be A-B-C-D**! Tip: mark your answers on this exam **first**, and only after you are certain of your answers, copy them to the multiple-choice answer form.
- Read every question properly and in the case of the open questions, give **all information** requested, which should always include a brief explanation of your answer. Do not however give irrelevant information – this could lead to a deduction of points.
- Note that the minimum score per (sub)question is 0 points.
- You may write on this exam paper and take it home.
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Open questions:

Question:	17	18	19	20	21	22	23	24	25	Total:
Points:	6	6	5	9	6	10	5	6	12	65

Multiple-Choice questions

1. Consider the formal structure S with domain $D^S = \mathbb{N}$, and truth sets $P^S = \{0, 1, 3, 4\}$, $Q = \{5, 7, 9, 11\}$, and $R^S = \{(0, 823), (23, 127)\}$.
   Which of the following statements is **true** for the structure S? Note carefully the order in which x and y are introduced and used in R!
   A. $\forall x(P(x) \to \exists y((y < x \lor Q(y)) \land (R(y, x))))$
   B. $\forall y(P(y) \to \exists x((y < x \land Q(x)) \lor (R(x, y))))$
   C. $\forall x(P(x) \to (\exists y((y < x \land Q(y))) \lor \exists y(R(y, x))))$
   D. $\forall y(P(y) \to (\exists x((y < x \land Q(x))) \land \exists x(R(x, y))))$

2. Which of the following is **true** when we want to prove a property $Q(x)$ is true for all statements x in PROP?
   A. In the induction step we prove that $\forall x \in \text{PROP}(Q(x) \to Q(\neg x))$.
   B. In the induction step we prove that $\forall x(x \in \text{PROP} \to \neg x \in \text{PROP})$.
   C. In the induction step we prove that $\forall x(Q(x) \in \text{PROP} \to Q(\neg x) \in \text{PROP})$.
   D. In the induction step we prove that if $Q(p_i)$ for a propositional variable p_i, then also $Q(\neg p_i)$.

3. Which of the following is **true** about all sets A?
   A. If $\emptyset \in A$, then there is a set B, such that $\mathscr{P}(B) = A$.
   B. If $C \subseteq A$, then there is a set B such that $B \in \mathscr{P}(A)$ and $B \subseteq C$.
   C. If $|A|$ is divisible by 4, then there is a set B, such that $\mathscr{P}(B) = A$.
   D. If there is a set B such that for all $x \in A$: $x \subseteq B$, then $\mathscr{P}(B) = A$.

4. On the multiple choice test of this course, in question 6 a mistake in the answers argued that $P \to Q \equiv Q \to P$ is not sufficient for $P \leftrightarrow Q$. Why is this a mistake?
   A. Because $(P \to Q) \leftrightarrow (Q \to P)$ is a tautology.
   B. Because $(P \to Q) \leftrightarrow (Q \to P)$ is a contradiction.
   C. Because $((P \to Q) \leftrightarrow (Q \to P)) \to (P \land \neg Q)$ is a contradiction.
   D. Because $((P \to Q) \leftrightarrow (Q \to P)) \to (\neg P \lor Q)$ is a contingency.

5. Which of the following statements is **true**?
   A. $\mathbb{R} \subseteq \mathbb{Q}$
   B. $\mathbb{Q} \subseteq (\mathbb{N} \Delta \mathbb{Z})$
   C. $(\mathbb{Z} \setminus \mathbb{Q}) \subseteq \mathbb{N}$
   D. $(\mathbb{Q} \setminus \mathbb{Z}) \subseteq \mathbb{N}$

6. Which of the following statements is **true** about an arbitrary statement A?
   A. A is satisfiable iff A is valid.
   B. A is valid iff $\neg A$ is not valid.
   C. A is valid iff $\neg A$ is not satisfiable.
   D. A is satisfiable iff $\neg A$ is not satisfiable.

7. Someone wants to combine a proof by division into cases with a proof by induction to prove the property $P(n)$ for all integers $n \geq 6$. As a result her induction step does the following: "Take arbitrary k, such that $P(k)$ holds. If k is even: then we show $P(k + 1)$ holds, if k is odd, then we show $P(k + 3)$ holds." Which of the following should they prove in their base case to make the proof valid?
   A. $P(0) \land P(1)$
   B. $P(1) \land P(4)$
   C. $P(2) \land P(5)$
   D. $P(2) \land P(6)$

8. Let $f(P)$ be the number of ones in the column for the main connective of the compound proposition P, and let $a(P)$ be the number of unique propositional variables in P.
Which of the following statements is **true**?
   A. If $a(P) = a(Q)$ then $f(P) = f(Q)$.
   B. If $a(P) > a(Q)$, then $f(P \wedge Q) > f(P \vee Q)$.
   C. If $a(P) = a(P \wedge Q)$, then $f(P) \geq f(P \wedge Q)$.
   D. If $a(P) < a(P \vee Q)$, then $f(P) \geq f(P \vee Q)$.

9. Consider the recursively defined set $A \subseteq \mathbb{Z}$ using the rules:
   I. $13 \in A$ and $3 \in A$
   II. $x \in A \rightarrow x - 12 \in A$
   III. $(3x \in A \wedge x \in \mathbb{Z}) \rightarrow x \in A$
   IV. Nothing other than created by the rules above is in A.

   Which of the following is **true**?
   A. $-23 \notin A$
   B. $0 \in A$
   C. $2 \in A$
   D. $25 \notin A$

10. Someone argues that if we add the rule: $x, y \in A \rightarrow 2x + y \in A$ to the set of rules above, then $A = \mathbb{Z}$. Is this correct?
   A. Yes
   B. No, as $0 \notin A$
   C. No, as $0.5 \notin A$
   D. No, as $5 \notin A$

11. We say a set A is bounded iff $\exists x, y \in \mathbb{R}(\forall a \in A(x < a < y))$. Which of the following statements is **true**?
   A. $\mathbb{N}$ is bounded.
   B. There are finite sets that are **not** bounded.
   C. There are infinitely many bounded infinite sets.
   D. The intersection of a bounded set and an unbounded set is always finite.

12. Which of the following relations is a well-defined function $f : A \rightarrow B$ with $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.
   A. $L = \emptyset$
   B. $I = \{(a, 1), (a, 2), (a, 3)\}$
   C. $N = \{(a, 1), (b, 1), (c, 1)\}$
   D. $K = \{(1, a), (2, b), (3, c)\}$

13. Which of the following statements is **true**?
   A. If $f : A \rightarrow B$ and $A = B$, then f is injective.
   B. If $|A| \geq |B|$, then $f : A \rightarrow B$ **cannot** be injective.
   C. If $f : A \rightarrow B$ is injective and surjective, then $A = B$.
   D. If $f : A \rightarrow B$ is surjective and $|A| \geq |B|$, then f is injective.

14. Consider the following propositions:

   - p represents: "(NPC ∩ P) = ∅"
   - q represents: "P = NP"
   - r represents: "You lose all your bitcoin."
   - s represents: "You give out your password."

   Which of the following statements accurately describes the following:
   "If (NPC ∩ P) ≠ ∅, then P = NP. If you give out your password or if P = NP, then you lose all your bitcoin."

   A. $(p \to q) \land ((\neg q \lor s) \to r)$
   B. $(\neg p \to q) \land ((q \lor s) \to r)$
   C. $\neg p \to (\neg q \land ((q \lor s) \to r))$
   D. $\neg(p \to q) \land (\neg(q \lor s) \to r)$

15. After their many adventures stepping on K-Maps and converting speeds, the Ducks and Sharks have accomplished their mission. On their way home however, the newly created friendship between Donald McDuck and Shirley McShark threatens to break down. They disagree about the notion of equivalence classes. The argument is sparked by the following diagram they find in some street art down at the sea, depicted in Figure 1.

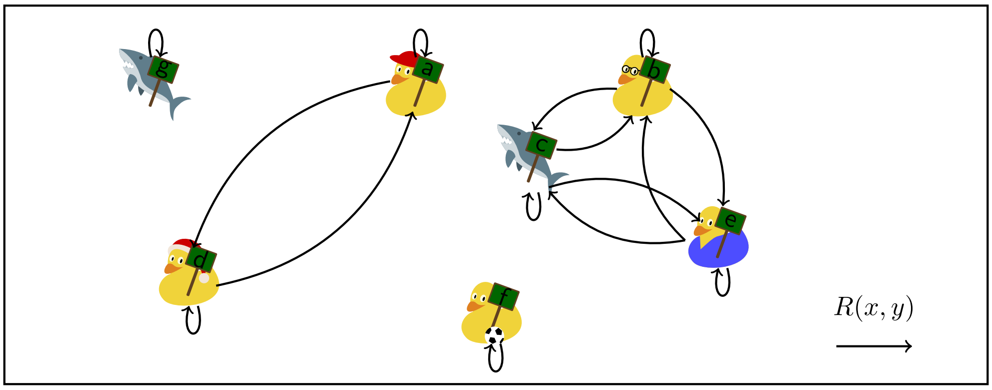

Figure 1: The street art by the sea

   Which of the following statements is true about the relation R?

   A. $[a] = [f]$ if we just add two elements to R.
   B. $[g] = [f]$ if we just add two elements to R.
   C. We can half the number of partitions by just adding two elements to R.
   D. We can double the number of partitions by just removing two elements from R.

16. Which of the following is **true**?

   A. If $A \subset C$, then $A \times B \subseteq C \times B$.
   B. If $A \subseteq B$, then $A \times B \subset B \times B$.
   C. If $A = B$, then $A \times B \times A = (B \times A) \times B$.
   D. If $A = B$, then $A \times B \times A \subseteq B \times (A \times B)$.

Open questions

17. (a) (1 point) Give a Venn Diagram for the set $(A - B) \cup (B - C)$.

    (b) (5 points) Claim: For all sets A if $|A|$ is odd and $\emptyset \notin A$ then there is **no** set B such that $\mathscr{P}(B) = A$. If the claim is true, prove it. If it is false, give a counterexample.

18. (6 points) Prove that for all integers $n \geq 1$: $7 \mid 2^{n+2} + 3^{2n+1}$. Make sure to show your intermediate steps.

19. (a) (2 points) What are the three properties an equivalence relation needs to fulfil? Give both the name and a brief description (or logically written definition) of the properties.

    (b) (3 points) For following relation describe for each of the properties you gave us in question a, whether the relation satisfies that property and *why* it does (not). Two ducks a and b are in the relation R if a has more children than b.

20. (a) (2 points) Consider: $(p \to q) \therefore (q \to p)$. If this argument is valid, prove it. If it is not give a counterexample and explanation to disprove it.

    (b) (3 points) Consider: $(\exists x(P(x)) \wedge \exists y(Q(y))) \leftrightarrow \forall x(P(x) \wedge Q(x))$. If this is satisfiable, prove it. If it is not, explain why not.

    (c) (4 points) Give a counterexample and explanation as to why it is a counterexample for the statement: for all sets A, B, C it holds that $(((A - B) \cap (C \cup B)) \subseteq B) \to A \cap B = \emptyset$.
    Note that a Venn Diagram does not constitute a counterexample!

21. Consider the sets $A = \{a, b, \{c, d\}\}$ and $B = \{1, 2, 3, 4, \{5, 6\}\}$.

    (a) (2 points) Give the powerset of A.

    (b) (2 points) Give a function $f : A \to B$.

    (c) (2 points) If your function f has an inverse, give it. If it does not, explain why not.

22. (10 points) Consider the following function:

```
function Foo(A,B)
    x ← A
    y ← B
    p = 0
    while y ≠ 0 do
        if 2 | y then
            x ← 2 · x
            y ← y/2
        else
            p ← p + x
            y ← y − 1
        end if
    end while
    return p
end function
```

The ancient Egyptians already used this algorithm to compute $A \cdot B$ for $A, B \in \mathbb{N}$. Prove that this algorithm indeed computes the multiplication. Hint: You can prove $xy + p = AB$ to be a useful invariant.

23. (5 points) Consider the set $A = \{x \in \mathbb{N} \mid \exists y \in \mathbb{Z}(x = 4y)\}$.
Claim: Set A has the same cardinality as $\mathbb{Q}$.
If this claim is true, prove it. If this claim is false, disprove it. Start you answer with: "True" if you believe the claim to be true and with "False" if you believe the claim to be false.

Final questions are on the back!

24. (a) (2 points) Create a truth table for $(p \wedge \neg q) \leftrightarrow \neg(q \vee p)$.

    (b) (4 points) Consider a new normal form similar to DNF and CNF. The Implicative Normal Form (INF) requires propositions to be an implication (just the one!) of disjuctions. Similar to DNF and CNF, negations may only occur in front of literals, not on compound propositions. Examples of expressions in INF include $p \rightarrow q$, $p \vee q$, and $(p \vee q \vee z) \rightarrow (q \vee r)$.
    Rewrite $(p \vee \neg q) \vee \neg(r \rightarrow p)$ to INF, simplify your answer as much as possible.

25. (a) (8 points) Consider the recursive definition of the set A:

    I. $3 \in A$, $15 \in A$
    II. $x \in A \rightarrow 8x + 24 \in A$
    III. $x, y \in A \rightarrow 2x - 7y \in A$
    IV. Nothing other than created by the rules above is in A.

    Prove that every number in A is divisible by 3.

    (b) (4 points) Create **a recursive definition** for a set $A \subseteq \mathbb{Z}$ that contains all numbers divisible by 11.

End of the exam