Resit Reasoning and Logic (CSE1300)

Exam created by Stefan Hugtenburg & Neil Yorke-Smith

Please read the following information carefully!

- This exam consists of 16 multiple-choice questions and 10 open questions.
- The points for the multiple-choice part of the exam are computed as $1 + 9 \cdot \max(0, \frac{\text{score} - 0.25 * 16}{0.75 * 16})$. This accounts for a 25% guessing correction, corresponding to the four-choice questions we use.
- The grade for the open questions is computed as: $1 + 9 \cdot \frac{\text{score}}{61}$.
- The **final grade for the exam** is computed as: $0.4 \cdot MC + 0.6 \cdot Open$.
- This exam corresponds to all non-starred sections of the book: *Delftse Foundations of Computation* (version 1.1).
- You have 3 hours to complete this exam.
- Before you hand in your answers, check that the sheet contains your name and student number, both in the human and computer-readable formats.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Tip: mark your answers on this exam **first**, and only after you are certain of your answers, copy them to the multiple-choice answer form.
- Read every question properly and in the case of the open questions, give **all information** requested, which should always include a brief explanation of your answer. Do not however give irrelevant information – this could lead to a deduction of points.
- Note that the minimum score per (sub)question is 0 points.
- You may write on this exam paper and take it home.
-

Open questions:

Question:	17	18	19	20	21	22	23	24	25	26	Total:
Points:	6	6	4	7	5	6	8	5	4	10	61

Multiple-Choice questions

1. Which of the following claims is **true**?
   - A. There is a relation that is transitive, yet not symmetric.
   - B. There is no relation that is not transitive, not reflexive, and not symmetric.
   - C. The relation $R = \emptyset$ over $\mathbb{N} \times \mathbb{N}$ is both reflexive and transitive.
   - D. The relation $R = \mathbb{N} \times \mathbb{N}$ over $\mathbb{Z} \times \mathbb{Z}$ is both reflexive and transitive.
2. Which of the following constitutes a counterexample for the claim: $\forall x \in \mathbb{Z}(\exists y \in \mathbb{N}(x^2 + 3x - 1 > y))$?
   - A. Take $x = 0$ and $y = 8$.
   - B. Take $x = 0$ and an arbitrary y.
   - C. Take an arbitrary x and take $y = x + 3$.
   - D. Take an arbitrary x and take $y = x - 10$.
3. Given a domain $D = \{a, b, c, d\}$. Which of the following statements is **not** satisfiable in this domain?
   - A. $P(a)$
   - B. $\exists x(P(x) \vee Q(x))$
   - C. $P(a) \wedge \forall x \forall y(\neg P(x) \wedge R(x, y))$
   - D. $P(a) \rightarrow \forall x \forall y(\neg Q(x) \wedge \neg P(y))$
4. The ducks are at it again, claiming they have invented a new type of proof. They are proud to present to the world: proofs by inducktion. The induktion proof method works over $\mathbb{Q}$. Similar to a mathematical induction proof, it has a base case, but now it also has an inducktive step.

   The ducks want to prove all elements $q \in \mathbb{Q}$ larger than or equal to 0 have a property $P(q)$. In the base case they prove that $P(0)$ holds. Now in the inducktion step they intend to show that for all integers $a, b \in \mathbb{Z}$: $P\left(\frac{a}{b}\right) \rightarrow P\left(\frac{a+1}{b+1}\right)$ holds.

   Which of these statements is **true**?
   - A. This proof method is valid.
   - B. The base case should also include $P\left(\frac{1}{1}\right)$ and $P\left(\frac{1}{2}\right)$.
   - C. The inducktion step should read: $P\left(\frac{a}{b}\right) \rightarrow \left(P\left(\frac{a+1}{b}\right) \wedge P\left(\frac{a}{b+1}\right)\right)$.
   - D. The inducktion step should read: $P\left(\frac{a}{b}\right) \rightarrow \left(P\left(\frac{2*a}{b}\right) \wedge P\left(\frac{a}{b*2}\right)\right)$
5. Someone wants to prove the following claim using a proof by contradiction.
   Claim: for all integers $n \geq 0$, if n is a prime number larger than 2, then there is a prime number k such that $k = 3n + 4$.
   Which of the following should be the assumption to start the proof by contradiction?
   - A. Assume that for an arbitrary integer $n > 2$ there exists a prime number k such that $k \neq 3n+4$.
   - B. Assume that there is no integer $n > 2$ such that there is also a prime number k with $k = 3n+4$.
   - C. Assume that for an arbitrary integer $n > 2$ it holds that all prime numbers k can be written as $k = 3n + 4$.
   - D. Assume there is an integer $n > 2$ such that for all prime numbers k it holds that $k \neq 3n + 4$.
6. Consider a bijective function $f : \{0, 1, 2, 3, 4\} \rightarrow B$. Which of the following sets **could** be the set B?
   - A. $\emptyset$
   - B. $\{1, 2, 3\}$
   - C. $\{1, 2, 3, 4, 5\}$
   - D. $\mathbb{N}$

7. Which of the following is **true**?
   A. $|\mathscr{P}(A)| \in B \rightarrow |B| \geq 2^{|A|}$
   B. $\mathscr{P}(A) \subseteq B \rightarrow A \in B$
   C. $\emptyset = \{\emptyset\}$
   D. $\mathscr{P}(\emptyset) = \mathscr{P}(\mathscr{P}(\emptyset))$

8. Someone is investigating the set $\mathbb{S}$, a completely new set of numbers never studied before. (If you think you have heard of the set $\mathbb{S}$ before, that is not this set. This set exists only for the purpose of this exercise.) The creator of this set is proud to announce the following is all true: $\mathbb{Z} \subseteq \mathbb{S}$, $-0.25 \in \mathbb{S}$ and there is a bijection from $\mathbb{N}$ to $\mathbb{S}$.

   Which of the following statements is now **true**?
   A. $\mathbb{S}$ contains less elements than $\mathbb{Z}$, as some element in $\mathbb{Z}$ is not in $\mathbb{S}$.
   B. $\mathbb{S}$ contains more elements than $\mathbb{Z}$, as $-0.25 \notin \mathbb{Z}$.
   C. $\mathbb{S}$ is uncountably infinite, as it contains more elements than $\mathbb{Z}$.
   D. $\mathbb{S}$ and $\mathbb{Z}$ are both the same size, as they are both countably infinite.

9. Mia constructed a function f that given a positive number x returns all numbers a such that squaring a results in x. Which of the following **can represent** that function f?
   A. $f : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$, with $f(x) = (\sqrt{x}, -\sqrt{x})$
   B. $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Z}$, with $f(x) = (\sqrt{x}, -\sqrt{x})$
   C. $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, with $f(x, y) = (x^2, y^2)$
   D. $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, with $f(x, y) : x^2 = y$

10. Given an argument A with premises $p_1, \ldots, p_n$ and conclusion c. Which of the following statements is **true**?
    A. If c is **not** satisfiable, then A is invalid.
    B. If $p_1 \wedge \ldots p_n$ is **not** satisfiable, then A is valid.
    C. A is valid iff $p_1 \wedge \cdots \wedge p_n \wedge c$ is satisfiable.
    D. A is valid iff $(p_1 \wedge \cdots \wedge p_n) \rightarrow c$ is satisfiable.

11. Consider $A = \{x \in \mathbb{N} \mid -8 \leq x \leq 17\}, B = \{0, 1, 2, 3, 4\}$. Which of the following is **true**?
    A. $A \times B \subseteq B \times A$
    B. $|\{(-1, 1)\} \times B \times A| = 1$
    C. $\{(x, y) \mid x \geq 0 \wedge y \geq 2\} \subseteq A \times B$
    D. $\{(x, y) \mid x, y \in \mathbb{N} \wedge x \leq -4 \wedge y \geq 2\} \subseteq A \times B$

12. For the OOP-Project Sander needs to somehow divide students over many different number of groups. For this year he expects to need about 80 of them! Someone argues that we should thus create a binary relation R, such that two students x, y have the relation R iff they are in the same OOP-P group. Since all students want to do the project, every student should be put in exactly one group. Which of the following properties must this relation R have?
    A. $\forall x, y(R(x, y))$ should hold.
    B. R should be an equivalence relation.
    C. R should be reflexive and symmetric, but not transitive.
    D. R should be reflexive and transitive, but not symmetric.

13. With the groups created Sander is now faced with the challenge of assigning TAs to groups. Every group should get exactly one TA and every TA should get more than one group. If A is the set of groups and B is the set of TAs, which of the following would be a valid description of a function f to assign TAs to groups?
    A. A bijective function $f : A \rightarrow B$
    B. An injective function $f : B \rightarrow \mathscr{P}(A)$
    C. A surjective function $f : \mathscr{P}(A) \rightarrow \mathscr{P}(B)$
    D. A bijective function $f : \mathscr{P}(A) \rightarrow B$

14. Consider the following predicates and constants:

- s for the Shortest Path Problem
- h for the Hamiltonian Path Problem
- t for the Travelling Salesman Problem
- v for the Vertex Cover Problem
- $Poly(x)$ means x is solvable in polynomial time
- $NPoly(x)$ means x is verifiable in polynomial time
- $I(i, x)$ means i is an instance of x.
- $O(i, d)$ means there is an algorithm that solves i in $O(|i|^d)$ time.

Given that we know the following conditions are all true:

1. $Poly(s)$
2. $NPoly(h)$
3. $NPoly(t)$
4. $P = \{x \mid Poly(x)\}$
5. $NP = \{x \mid NPoly(x)\}$
6. $(\exists y \forall z(I(z, h) \to (O(z, y) \land y \geq 0))) \to P = NP$

Which of the following statements is **guaranteed to be true** given these conditions?

A. If there is an instance of the Hamiltonian Path Problem that **cannot** be solved in $O(1)$ time, then $\neg Poly(t)$

B. If there is an instance of the Hamiltonian Path Problem that **cannot** be solved in $O(1)$ time, then $\neg NPoly(s)$

C. If all instances of the Hamiltonian Path Problem can be solved in $O(1)$ time, then $t \in P$.

D. If all instances of the Hamiltonian Path Problem can be solved in $O(1)$ time, then $\forall x(Poly(x) \land NPoly(x))$.

15. Consider again the constants and predicates from question 14. For this question, consider **only** the following criteria:

1. $\forall x(I(x, s) \to I(x, t))$
2. $\forall x(I(x, t) \to I(x, v))$

Which of these statements is **true**?

A. x being an instance of the Vertex Cover Problem is necessary for x being an instance of the Shortest Path Problem.

B. x being an instance of the Shortest Path Problem is necessary for x being an instance of the Travelling Salesman Problem.

C. x being an instance of the Vertex Cover Problem is sufficient for x being an instance of the Shortest Path Problem.

D. x being an instance of the Travelling Salesman Problem is sufficient for x being an instance of the Shortest Path Problem.

16. Consider again the constants and predicates from question 14. For this question, consider **only** the following criteria:

1. $P = \{x \mid Poly(x)\}$
2. $NP = \{x \mid NPoly(x)\}$

Which of these statements is **guaranteed to be true**?

A. $P \subseteq NP \leftrightarrow \forall x(NPoly(x) \to Poly(x))$

B. $P \times NP = \emptyset \leftrightarrow \forall x(Poly(x) \land \neg NPoly(x))$.

C. $P \cap NP = \emptyset \leftrightarrow \forall x(Poly(x) \leftrightarrow \neg NPoly(x))$

D. $P \cup NP = \emptyset \leftrightarrow \forall x(\neg Poly(x) \lor \neg NPoly(x))$

Open questions

17. (a) (3 points) Consider a new type of operator that operates on three propositions at a time. This ternary operator is defined by the following truth table:

a	b	c	$a \xrightarrow{b} c$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Construct a truth table for $q \xrightarrow{r} \neg(p \vee r)$.

(b) (1 point) We can model truth tables also as Venn diagrams. Given a proposition with three propositional variables p, q, r we draw a Venn Diagram with three sets A, B, C. We colour an area in the Venn diagram if and only if the corresponding row in the truth table is a 1. For p we take the area in A, for q the area in B and for r the area in C. Thus for the row $p = r = 0, q = 1$, we should consider the area that is in A and in C, but not in B.

Draw a Venn diagram that models the truth table for $p \xrightarrow{q} r$.

(c) (2 points) Rewrite the following to DNF: $(p \rightarrow q) \wedge \neg(p \vee q)$. Simplify your answer as much as possible and show your intermediate steps.

18. (6 points) Prove that for all integers $n \geq 2 : \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$. Make sure to show your intermediate steps.

19. (4 points) Give a *recursive* definition of the set S that contains only the numbers 8 and 13, all numbers that are equal to 4 times an element of S, as well as numbers that are exactly 3 away from a number in S.

20. (a) (3 points) Prove the following claim:
for all sets A, B, C it holds that $(A \in B \wedge B \cap C = \emptyset) \rightarrow A \notin C$

(b) (4 points) Consider the following claim:
for all sets A, B, C it holds that $(B \in \mathscr{P}(A) \wedge C \in A) \rightarrow (C \in A \Delta B)$. Give a counterexample with concrete sets to prove this claim is false. Also explain how your counterexample shows the claim is false.

21. Given the sets $A = \{(1, 2), 3\}$ and $B = \{a, b\}$ for the universe $U = \{(1, 2), 3, a, b, c, d, \textit{phoenix}, \textit{duck}\}$.

(a) (1 point) Give the set A^c.

(b) (2 points) Give the set $A \times B$.

(c) (2 points) Give a surjective function $f : \mathscr{P}(A) \rightarrow A$

22. (6 points) Prove the following claim for all integers $z \in \mathbb{Z}$: if 4 does not divide z^2 then z is odd.

23. (a) (2 points) Consider the statement "all humans love ducks". Give an example of an argument that cannot be expressed in propositional logic that uses this statement.

(b) (3 points) Give an example of a set $R \subseteq \mathbb{N} \times \mathbb{N}$ that is a binary relation but not a function $f : \mathbb{N} \rightarrow \mathbb{N}$. Explain clearly why it is not a function.

(c) (3 points) Give an example of a valid argument that has as its conclusion: "All dogs love Marmite[1] and all dogs do not love Marmite". Explain why your argument is valid.

[1]Marmite is a spread, with the marketing slogan: "Love it or hate it".

24. For this question you need to translate claims from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.

    (a) (2 points) Maarten likes all ducks

    (b) (3 points) Stefan fries an oliebol[2], if he owns a kitchen.

25. (4 points) If the following formal argument is valid, explain why in at most 5 lines. If it is not, prove this using a formal structure with a domain of at most 5 elements. Start your answer with "valid" or "invalid".

    $$\begin{array}{l} P(s) \land (Q(t) \rightarrow \exists z(R(s,z))) \\ \forall x \exists y(P(x) \lor Q(y)) \\ \forall x(P(x) \leftrightarrow \exists z(R(x,z))) \\ \hline \therefore \exists y(Q(y)) \end{array}$$

26. (10 points) Consider the following algorithm that computes the rounded down value of a division. For example for `FOO(5,3)` it will return $\lfloor 5/3 \rfloor = 1$.

    function FOO(A,B)
        $x \leftarrow A$
        $y \leftarrow B$
        $q = 0$
        while $x \geq y$ **do**
            $x \leftarrow x - y$
            $q \leftarrow q + 1$
        end while
        return q
    end function

    Prove that this algorithm indeed computes this rounded down division when given positive integers A and B, that is it returns the largest integer z such that $Bz \leq A$. Hint: You can prove $x = A - qy \land x \geq 0$ to be a useful invariant.

[2] A Dutch fatty snack often consumed around New Year. Note that there are many oliebollen in the world.