NOTE: This exam was never conducted on paper, but digitally! As a result some questions have been formulated slightly differently, the layout of this PDF is not optimised, and other issues may appear that were not present in the digital version of this exam.

Additionally the midterm was formative in the year of this endterm, meaning more questions were asked here about the midterm material.

1. (5 points) • **Variant Kokofruit:**
   For this question you need to translate claims from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.
   (a) – **Variant 2**For this question you need to translate a Kokofruit grows on tall kokotrees from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.
   Kokofruit grows on tall kokotrees
   (b) – **Variant 1**For this question you need to translate a If you save some trunk, you have a floor. from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.
   If you save some trunk, you have a floor.

2. (4 points) • **Variant WILTY:**
   For this question you need to translate claims from predicate logic to natural language. We use the following predicates and constants:
   - $d$ is for David
   - $l$ is for Lee
   - $v$ is for Victoria
   - $Panelist(x)$ for $x$ is a Panelist
   - $Lollypopman(x)$ for $x$ is a lollypop man (someone who helps children cross the street)
   - $Accuses(x, y)$ for $x$ accuses $y$ of lying
   - $IsCaptainOf(x, y, z)$ for $x$ is captain of team with other members $y$ and $z$

   These predicates and constants are inspired by the tv-show Would I lie to you?
   (a) – **Variant 2**For this question you need to translate a claim from predicate logic to natural language. We use the following predicates and constants:
   * $d$ is for David
   * $l$ is for Lee
   * $v$ is for Victoria
   * $Panelist(x)$ for $x$ is a Panelist
   * $Lollypopman(x)$ for $x$ is a lollypop man (someone who helps children cross the street)
   * $Accuses(x, y)$ for $x$ accuses $y$ of lying
   * $IsCaptainOf(x, y, z)$ for $x$ is captain of team with other members $y$ and $z$

   These predicates and constants are inspired by the tv-show Would I lie to you?
   The claim: $\exists x(Lollypopman(x) \land x \neq d) \rightarrow Panellist(l)$
   (b) – **Variant 0**For this question you need to translate a claim from predicate logic to natural language. We use the following predicates and constants:
   * $d$ is for David
   * $l$ is for Lee
   * $v$ is for Victoria
   * $Panelist(x)$ for $x$ is a Panelist
   * $Lollypopman(x)$ for $x$ is a lollypop man (someone who helps children cross the street)
   * $Accuses(x, y)$ for $x$ accuses $y$ of lying
   * $IsCaptainOf(x, y, z)$ for $x$ is captain of team with other members $y$ and $z$

   These predicates and constants are inspired by the tv-show Would I lie to you?
   The claim: $\forall x \forall y \forall z(Captain(x, y, z) \rightarrow (x = d \lor x = l))$

3. (7 points) Note that the arguments look a bit weird in the PDF, this is so they look better in WebLab (hopefully)

   1. **Variant 7:**
   Consider the following quaternary operator $p \overset{q}{\underset{r}{\rightarrow}} s$, with the truth table:

| p | q | r | s | $p \xrightarrow[r]{q} s$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

Consider now the following argument:

$$\frac{r \xrightarrow[(p \vee \neg q)]{r \leftrightarrow p} q}{\therefore p \xrightarrow[p]{q} q}$$

- For 6 points create a single truth table containing the truth values of the premise and of the conclusion.
- For 1 point explain how your truth table shows the argument is (in)valid, refer to specific lines from your truth table in your explanation.

You may write the quaternary operator as p - (q,r) -> s or in LaTeX.

For your convenience, here is also a version of the argument in LaTeX (please note that in TeX the square brackets for the xrightarrow indicate what goes below, and the curly braces what goes above the arrow):

```
\\(r \xrightarrow[(p \vee \neg q)]{r \leftrightarrow p} q\\)
\\(\therefore p \xrightarrow[p]{q} q\\)
```

For a slightly larger table, you can copy/paste this. Please note that this table might be too big or still too small! You may remove unused columns/rows or simply leave them empty.

```
| col 1  | col 2 | col 3 | col 4 | col 5 | col 6 | col 7 | col 8 | col 9 | col 10 | col 11 |
|:------:|:-----:|:-----:|:-----:|:-----:|:-----:|:-----:|:-----:|:-----:|:------:|:------:|:
| row 1  |       |       |       |       |       |       |       |       |        |        |
| row 2  |       |       |       |       |       |       |       |       |        |        |
| row 3  |       |       |       |       |       |       |       |       |        |        |
| row 4  |       |       |       |       |       |       |       |       |        |        |
| row 4  |       |       |       |       |       |       |       |       |        |        |
| row 5  |       |       |       |       |       |       |       |       |        |        |
| row 6  |       |       |       |       |       |       |       |       |        |        |
| row 7  |       |       |       |       |       |       |       |       |        |        |
| row 8  |       |       |       |       |       |       |       |       |        |        |
| row 9  |       |       |       |       |       |       |       |       |        |        |
| row 10 |       |       |       |       |       |       |       |       |        |        |
| row 11 |       |       |       |       |       |       |       |       |        |        |
| row 12 |       |       |       |       |       |       |       |       |        |        |
| row 13 |       |       |       |       |       |       |       |       |        |        |
| row 14 |       |       |       |       |       |       |       |       |        |        |
| row 15 |       |       |       |       |       |       |       |       |        |        |
| row 16 |       |       |       |       |       |       |       |       |        |        |
```

4. (2 points) • **Variant 6:**

For one point give a valid argument whose conclusion is: "there are no surjective functions".

We say an argument is sound if it is valid and if its premises are also true. For one point explain whether or not your argument is sound.

5. (a) (3 points) • **Var7**

If the following set of statements is satisfiable prove it with a structure with at least 3 and at most 5 elements in the domain. If it is not, explain why not.

– $\forall x \neg(\neg Q(x) \vee P(a))$
– $\exists y \forall z(\neg R(y, z) \rightarrow Q(z))$
– $(Q(c) \vee R(a, b))$

(b) (2 points) • **Var2**

Donna the coati queen and Marty the owl king are trying to resolve their differences, it is time to move on from the tragic events that turned them against each other all those years ago.

Unfortunately the inhabitants of their kingdoms have different ideas about this and are not quite ready to bury the hatchet. During the first of a series of what are supposed to be peaceful events, the coatis and owls gathered there have taken up arms and are ready to fight.

The figure below shows these creatures ready to fight each other! (Just to make sure, $a$ is a coati and $b$ is an owl).


In order to assuage the situation, you try to make sure the following claim is **true**:

$\forall x((Coati(x) \wedge CrystalBall(x)) \rightarrow \exists y(Owl(y) \wedge LeftOf(x, y)))$

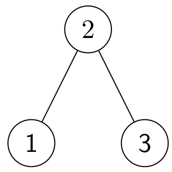
Is the claim already true? If so, explain why. If it is not, explain which coati or owl we should move to make the claim true and how that makes the claim true. Please note you can only move one owl or one coati!

6. (4 points) • **Variant 2** Consider the following recursively defined structure $T$ that represents a set of trees as used in computer science.

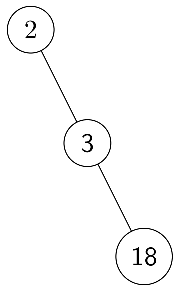
– $\emptyset \in T$
– $(T_1, T_2 \in T \wedge r \in \mathbb{N}) \rightarrow (T_1, r, T_2) \in T$

  – Nothing else is in $T$

You can visualise a tree like $((\emptyset, 1, \emptyset), 2, (\emptyset, 3, \emptyset))$ as follows:




You can visualise a tree like $(\emptyset, 2, (\emptyset, 3, (\emptyset, 18, \emptyset)))$ as follows:




Create a recursive function $f : T \to \mathbb{N}$ such that: f(t) gives the largest of the numbers in $t$, or 0 if no such value exists

7. (7 points) • **Variant 3:**

Prove the following claim with mathematical induction:

$\forall n \geq 4, n \in \mathbb{N} \quad \prod_{i=4}^{n} (\frac{i^2}{2} - i) = \frac{n! \cdot (n-2)!}{3 \cdot 2^{n-2}}$

For your convenience, in LaTeX:

```
$\forall n \geq 4,n \in \mathbb{N} \quad \prod_{i = 4}^{n} (\frac{i^2}{2} - i) = \frac{n! \co
```

Note that you may structure your equations in a list, something like:

```
- 1 + 1 =
- 1 + 1 - 2 + 2 =
- -1 + 3 =
- 2
```

8. (8 points) • **Var 5:**

Consider the following recursively defined set $S$ of words:

I. $neil, b \in S$

II. $x \in S \to azxaa \in S$

III. $x, y \in S \to zxbyba \in S$

IV. Nothing else is in $S$

For 1 point give an example of a word of length 5 that is in $S$. (Note that the length of a words is the number of letters in it. For example the length of the word "spoon" is 5.)

For 6 points, prove that $\forall w \in S$(twice the number of $z$'s in $w$ is at most as high as the number of $a$'s)

9. • **Variant 9:**

Given the sets $A = \{19, \text{impostor}, \text{marty}, b, d, u, x\}$, $B = \{9, \text{coati}, \text{maya}, b, d, o, u\}$, $C = \{9, \text{impostor}, \text{maya}, o, q, r$

give (the elements of) the set $(A \cup B) - C$.

(a) (3 points) • **Variant 3:**

Consider the following claim:

For all sets $A, B, C$: $((A \subseteq B) \wedge (\mathscr{P}(B) \cap C \neq \emptyset)) \to A \neq \emptyset$.

If this claim is true, prove it. If this claim is false, provide a counterexample and explain how your counterexample shows the claim is false.

For your convenience, here is the claim in LaTeX:

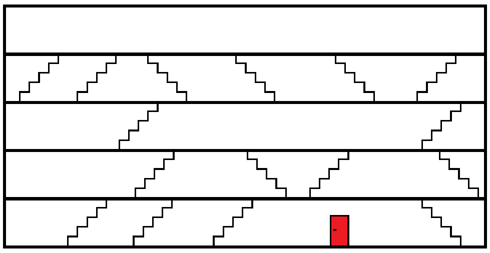
```
$((A \subseteq B) \wedge (\mathscr{P}(B) \cap C \neq \emptyset)) \to A \neq \emptyset$
```

10. (3 points) • **Variant 4:**
In a Tarski world, consider the relation $NotBelow(x, y)$ where $NotBelow(x, y)$ is true iff $x$ is not in a lower row than $y$. Describe if this is an equivalence relation by discussing the three properties of an equivalence relation and for each indicating why it does (not) hold.

11. (4 points) • **Variant 3:**
Given the set of pairs $(a, b)$ consisting of a natural number $a$ and a natural number $b$ which is at least 1 and lower or equal to the $(a + 1)^{th}$ odd natural number, formally given by:
$A = \{(a, b) | a \in \mathbb{N} \wedge b \in \mathbb{N} \wedge 1 \leq b \leq 2 \cdot a + 1\}$
Prove that $A$ is countably infinite by giving a bijection $A \to \mathbb{N}$ and explaining why this function is a bijection.
Hint: remember that the differences between each two subsequent squares are all the odd numbers $((x + 1)^2 - x^2 = 2x + 1)$.

12. (1 point) • **Variant 2** Show the following function is not well-defined:
$f : \mathbb{R} \to \mathbb{N}$, with $f(x) = x^2$

13. (4 points) • **Variant 0:**
In a building with multiple floors, you enter via the entrance at ground floor (the lowest floor). Then, you take a random staircase on that floor that you haven't taken before to another floor. You keep taking random staircases that you haven't taken before (either a staircase on your floor that leads to the floor above, or a staircase from the floor below that leads to the floor below) until you are at a floor where you have already taken all available staircases. It is given that each floor has an even amount of staircases (and the top one has no staircases). An example of a building is given below.



With a proof using an invariant, you can prove at which floor you will end up. The invariant is: each of the floors below you has an odd amount of staircases that you have walked, the floor where you are and each of the floors above you have an even amount of staircases that you have walked.
For 3 points: Show the invariant is an invariant, i.e. it is true before the loop and after every iteration of the loop.
For 1 point: Given this invariant, what floor do we end up on? Explain your answer (a formal proof is not required).

14. (3 points) • **Variant 6:**
Four young apprentices broke into a temple to steal four sacred element crystals. When the alarm went off, they panicked, and each of them swallowed the crystal they held right before they were caught. You must determine who ate which crystal. The elements compel their masters. Those who ate the earth and water crystals must speak the truth, while those who consumed fire and air must lie. The youths are too scared to confess their own transgressions. Instead, they fall to accusing each other. This is what they said:
   - Donna said something before you arrived (but you don't know what)
   - Coyote Caoti says that Coyote Caoti is telling the truth about that statement
   - Marty says that she didn't eat the water crystal
   - Dr. Whoo says that if Coyote Caoti ate the fire crystal, then Dr. Whoo ate the water crystal

Remember that:
   - Each student ate a different elemental crystal
   - The earth and water crystals force their owner to tell the truth
   - The fire and air crystals force their owners to lie

Who ate what crystal? Explain how you derived your answer.

End of the exam