Endterm Reasoning and Logic (CSE1300)

2021–22

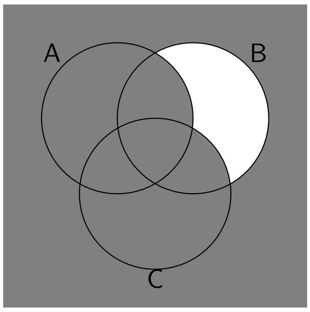
Please read the following information carefully!

- This exam consists of 11 open questions. The open questions are worth a total of 57 points and the points per open (sub) question are given in the (sub) question itself.
- The grade for this exam is computed as $1 + 9 \cdot \frac{\text{score}}{57}$. Note that you require a grade of at least 5 out of 10 to pass the course (assuming your average course grade is at least 5.75).
- This exam corresponds to all non-starred sections of the book: *Delftse Foundations of Computation* (version 2.0).
- You have 180 minutes to complete this exam.
- Before you hand in your answers, check that every sheet of answer paper contains your name and student number.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Read every question carefully and, in the case of the open questions, give **all information** requested. Do not however give irrelevant information: this could lead to a deduction of points.
- You may write on this exam paper and take it home.
- No marmots or hippos were harmed in the creation of this exam.
- Exam prepared by S. Hugtenburg, S. Dumančić, I. van Kreveld, N. Yorke-Smith
- Exam is ©2021 TU Delft.

Question:	1	2	3	4	5	6	7	8	9	10	11	Total:
Points:	5	5	3	5	4	6	5	6	5	8	5	57

Open questions

1. (a) (1 point) Which set is shown in **grey**? Give your answer in terms of A, B, and/or C.



   (b) (2 points) If the following claim is true, prove it. If it is false, provide a counterexample and a clear explanation detailing how the counterexample shows the claim is false. Start your answer with either the word *True*, or *False*.

   For all sets A, B, and non-empty sets C: $((A \cap B \neq \emptyset) \wedge (C \subset B)) \rightarrow A \cap C \neq \emptyset$

   (c) (2 points) Compute $\mathscr{P}(A)$ given that $A = \{(0,1), \emptyset, 2\}$.

2. (5 points) Given a bag with 10 white beans, 15 grey beans and 20 black beans. As long as there are at least 2 different colours of beans in the bag, you follow the following procedure.

   You take 2 beans of a different colour out of the bag, and you throw a bean of the other colour back in the bag. For example, when you would take a white and a black bean from the bag, you would throw a grey bean back in.

   Once there is only a single colour of beans left in the bag, you cannot continue the procedure, as you cannot take 2 different colour beans from the bag.

   If you try this procedure a couple of times, each time starting with 10 white beans, 15 grey beans and 20 black beans, you will notice that at the end of the procedure, there will always be the same colour of bean left! Which colour is it?

   Prove this fact using an invariant. In your proof, use the following invariant: the parity of the amount of white beans is the same as the parity of the amount of black beans, but different from the parity of the amount of grey beans. (Note that parity means whether a number is odd or even.)

3. Consider the following constants, predicates, and functions:

   - The domain of discourse is 'all objects' (including living beings).
   - $Marmot(x)$ means: x is a marmot
   - A is the set of all animals
   - a is for Amy, it is given that $a \in A$
   - m is for Maximillion (or just Max for short), it is given that $m \in A$.
   - $w : A \rightarrow \mathbb{N}$ is a function that gives the weight of an animal in kilograms.

   Translate the following phrases to natural language (English).

   (a) (1 point) $\exists B(B \in \mathscr{P}(A) \wedge \exists x(x \in B) \wedge \forall x \in B(w(x) > w(a)))$

   (b) (2 points) $\exists E(\forall x(Marmot(x) \leftrightarrow x \in E) \wedge \forall x(x \in E \rightarrow w(x) < 12))$

4. (a) (3 points) Rewrite $(p \rightarrow \neg(q \vee \neg r)) \rightarrow (p \wedge q)$ to CNF, simplify your result as much as possible. [Unsimplified answers can score at most 2 points.]

   (b) (2 points) Consider the following claim:

   "It is impossible for an implication and it's inverse to both be true at the same time."

   Is this claim true or false? Explain your answer.

5. Consider the following definition of a recursive structure:

   I. **End** $(x, \emptyset) \in A$ if $x \in \mathbb{N}$

   II. **Sum** $(a_1, +, a_2) \in A$ if $a_1, a_2 \in A$

   III. **Product** $(a_1, *, a_2) \in A$ if $a_1, a_2 \in A$

   IV. **Exclusivity** Nothing else is in A

   (a) (1 point) Give an example of an element $a \in A$ which you construct by applying rule I, II and III at least once each.

   (b) (3 points) Construct function $f : A \times \mathbb{N} \to A$ so that all odd values appearing in the structure are replaced with the second argument to f.

6. (a) (1 point) Imagine you want to prove the following claim $A \subseteq B$ by using a proof by contradiction. What should be the starting assumption of your proof? Your answer should start with a quantifier.

   (b) (3 points) Prove the following set of statements is satisfiable, in other words give sets A, B, C, such that the following statements all hold. Briefly explain why all statements hold.

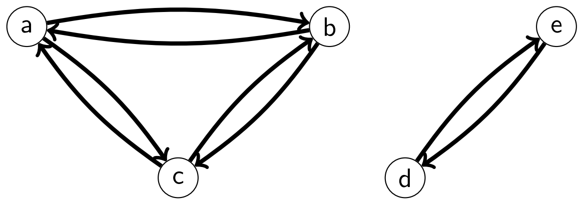
   - $|A|, |B|, |C| \leq 3$
   - $A \subseteq B$
   - $(B \cap \mathscr{P}(C)) \neq \emptyset$
   - $A \in \mathscr{P}(C)$

   (c) (2 points) Prove the following claim is false by providing a counterexample and a clear explanation as to why the counterexample shows the claim is invalid.
   $\forall n \in \mathbb{N}(4 \mid n \vee \exists m \in \mathbb{Z}(m \mid n \wedge m > 1))$

7. (a) (2 points) Draw a DAG $G = (V, E)$ such that the topological ordering is: (a, b, c, d, e, f) and $8 \leq |E| \leq 12$.

   (b) (2 points) A connected component C in a graph is a set of vertices such that from any vertex $c \in C$ there is a simple path to all other vertices in C. What are the connected components of the following graph $G_1 = (V_1, E_1)$? List them as sets of vertices.
   $V_1 = \{a, b, k, n, p, q, x, y, z\}$ $E_1 = \{\{x, n\}, \{a, k\}, \{k, z\}, \{p, x\}, \{z, y\}, \{y, a\}, \{n, b\}\}$

   (c) (1 point) For directed graphs we can consider E to be a relation on V^2. Now consider the following graph:



   If E is an equivalence relation, give $[a]$. Otherwise indicate clearly why E is not an equivalence relation.

8. (6 points) Consider an adapted version of our TREE definition, that defines full binary trees (where every node has either 0 or 2 children).

   1. **Leaf** $(x, \emptyset) \in FBTREE$ if $x \in D$
   2. **Internal** $(x, (t_1, t_2)) \in FBTREE$ if $x \in D$ and $t_1, t_2 \in FBTREE$
   3. **Exclusivity** Nothing else is in $FBTREE$

   Prove that for all $t \in FBTREE$ the number of internal nodes is exactly one less than the number of leaf nodes.

   You should use structural induction and should start by defining two functions to formalise the claim.

9. (5 points) Prove that $\forall n \in \mathbb{N}(3 \nmid 2n^2 + n + 1)$

10. (a) (2 points) Is the set $A = \{x\sqrt{2} \mid x \in \mathbb{Z}\}$ countably infinite? If so, give a bijection from $\mathbb{N}$ (there is no need to prove your function is bijection). If not, explain why not.

    (b) (2 points) Take the function $f : \mathbb{N} \to \mathbb{N}$ such that: $f(x) = \begin{cases} 1 & \text{if } x < 3 \\ 2f(x/2) - 3 & \text{if } 2 \mid x \\ 3f((x+1)/2) + 4 & \text{else} \end{cases}$

    Is f well-defined? If so, explain why. If not, give a clear example of an input and computation of f on that input that show f is not well-defined.

    (c) (2 points) A bijective function satisfies two properties. Give the names of both properties as well as a definition written in logic.

    (d) (2 points) Now consider the function $g : \mathbb{N} \to \mathbb{Z}$ such that $g(x) = x^2 + 3$. Explain for each of the properties from the previous question whether they hold or not.

11. (5 points) Prove the following claim is true using mathematical induction:

    For all integers $n \geq 2 : \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

End of the exam