# Resit Midterm Reasoning and Logic (CSE1300)

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**Please read the following information carefully!**

- This exam consists of 7 open questions. The open questions are worth a total of 60 points and the points per open (sub) question are given in the (sub) question itself.
- The grade for this exam is computed as $1 + 9 \cdot \frac{\text{score}}{60}$. Note that you require a grade of at least 5 out of 10 to pass the course (assuming your average course grade is at least 5.75).
- This exam corresponds to chapters 1 to 3 of the book *Delftse Foundations of Computation* (version 1.01), with the exception of the topic of induction.
- You have 90 minutes to complete this exam.
- Before you hand in your answers, check that every sheet of paper contains your name and student number, as well as the total number of sheets of paper that you hand in.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Read every question properly and in the case of the open questions, give **all information** requested. Do not however give irrelevant information, this could lead to a deduction of points.
- You may write on this exam paper and take it home.
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| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total: |
|---|---|---|---|---|---|---|---|---|
| Points: | 6 | 5 | 10 | 9 | 7 | 15 | 8 | 60 |

Learning goals coverage, based on the objectives of all lectures (strongly paraphrased):

| Goal | mt 17 | mt 18 | rmt 18 |
|---|---|---|---|
| translate logic to and from natural language | | 3,4 | 1,2 |
| describe $\wedge$,$\vee$,$\neg$,$\rightarrow$, and $\leftrightarrow$ operators | MC1 | | |
| construct a truth table | MC2, 11a | 1a,1b | 3a |
| determine prop. logic equivalence | 11b | | 3a |
| rewrite logical connectives | MC6 | | 3b |
| describe contrapositives, converses, and inverses. | MC3 | | |
| describe logic validity | | | |
| describe sufficient and necessary conditions | MC4 | | |
| prove validty of argument in prop. logic | MC5 | 1b | |
| describe the principle of explosion | | 1c | |
| explain why prop. logic is not sufficiently expressive | | | |
| describe $\forall$ and $\exists$ quantifiers | MC7,8 | 2c | |
| evaluate negation stmt. in pred. logic | | | |
| construct a Tarski's world | | | 4a |
| construct a formal structure in pred. logic | 12 | 2b | |
| evaluate claims about formal structures | 12 | 2a | 4b |
| construct counterexamples for claims | MC10 | 2a,2b,5c | 5b,bc |
| describe the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ | | | 5a |
| describe the form of a proof by division into cases | | 5b | 5c |
| describe the form of a proof by contradiction | | | |
| construct a proof by division into cases | | 7a | |
| construct a proof by contradiction | MC9 | | 6a |
| explain what a theorem prover is. | | | |
| describe the form of a proof by contrapositive | | 5a | |
| construct a proof by contrapositive | 13 | 7b | |
| describe the form of a proof by generalisation. | | 5a,5b | |
| construct a proof by generalisation | | | 6b |
| identify type of proof to use for a given claim | | 5b | |
| compute a sequence given a recursive definition | | 6a | 7a |
| construct and interpret recursive definitions | | 6b,6c | 7b |

1. For this question you need to translate claims from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.

   (a) (3 points) The BEP[1] is not part of the 2018 curriculum, but another project is.

   > **Answer:** $\neg PartOf(b, c) \land \exists p(Project(p) \land PartOf(p, c))$
   > Where $b$ is the BEP and $c$ is the 2018 curriculum.

   (b) (3 points) All magicians wear a silk hat or a rose.

   > **Answer:** $\forall x(Magician(x) \rightarrow \exists y(Wear(x, y) \land ((Silk(y) \land Hat(y)) \lor Rose(y))))$
   > The single-place predicates are defined as: $x$ is a ... and $Wear(x, y)$ is defined as $x$ wears $y$. Note that our answer allows $y$ to be both a silk hat and a rose. An alternative with xor is also fine.

[1] Bachelor Final Project

2. Translate the following two claims to natural language (English). The predicates and domain used in our claims are as follows:

- The domain of discourse is "all objects" (including living beings).
- $Genius(x)$ means: $x$ is a genius.
- $TimeMachine(x)$ means: $x$ is a time machine.
- $Owns(x, y)$ means: $x$ owns $y$.
- $d$ is "the Doctor" (this represents a single specific person).

(a) (2 points) $\forall x(Genius(x) \wedge \exists y(TimeMachine(y) \wedge Owns(x, y)))$

**Answer:** All objects are geniuses with a time machine.

(b) (3 points) $\forall x(Genius(x) \rightarrow (x = d)) \wedge \neg\exists y(TimeMachine(y) \wedge \neg\exists x(\textsf{Genius}(x) \wedge Owns(x, y)))$

**Answer:** Only the doctor is a genius and there is no time machine that is not owned by a genius. Alternatively: Only the doctor is a genius and he owns all the time machines.

3. (a) (7 points) Consider the new operator $\circ$ with the truth table:

| p | q | $(p \circ q)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Are $(p \to q) \circ (\neg r \leftrightarrow p)$ and $p \circ (\neg p \leftrightarrow r)$ equivalent? Explain your answer using a truth table.

**Answer:** Notice that $p \circ q \equiv q$. So for the truth table, only the right-most part is relevant. Notice also that for the second statement $q$ can be ignored regardless (but it can be ignored for both as the right-most part is only relevant).

| p | q | r | $(\neg r \leftrightarrow p)$ | $(\neg p \leftrightarrow r)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 |

As the truth tables are the same, they are equivalent.

(b) (3 points) Rewrite $(p \wedge q) \to \neg(r \vee q)$ to DNF. Simplify your result as much as possible.

**Answer:**

$$\begin{aligned}(p \wedge q) \to \neg(r \vee q) &\equiv \neg(p \wedge q) \vee \neg(r \vee q)\\ &\equiv \neg p \vee \neg q \vee (\neg r \wedge \neg q)\\ &\equiv \neg p \vee \neg q\end{aligned}$$

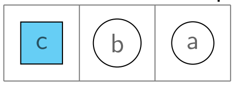
Alternative through K-map:

| p \ qr | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 |

4. (a) (4 points) Consider the following description of a Tarski World. There is one 'colour' in this world, which we call *filled* (use whatever your colour you are writing the exam in). Does an instance of a Tarski World exist with these properties? If so, give one with a domain of at most 5 elements. Make sure to label the objects with their name and that for filled objects these names are readable. If no such instance exists, explain why not in at most 5 lines.

   - $\forall x(Circle(x) \rightarrow \neg Filled(x))$
   - $\exists x(Circle(x)) \wedge \exists x(Filled(x))$
   - $RightOf(a, b)$
   - $LeftOf(a, b) \vee Square(c)$

   **Answer:** For example:

   Notice that at least one of $a$ or $b$ should be an empty circle. The other can be anything (but if it is a circle, it should be empty.)

   (b) (5 points) Consider the domain $\{a, b\}$ and the statements:
   $\forall x(P(x) \vee Q(x))$
   $\exists x \exists y(R(x, y))$
   $\forall x \forall y(R(x, y) \rightarrow (Q(x) \leftrightarrow P(x)))$

   Show that it is possible to create a structure $A$ in which all of the above statements hold, but $P^A \neq Q^A$. Explain why each of the claims hold. Answer in at most 10 lines.

   **Answer:** Take for example $P^A = \{a, b\}$, $Q^A = \{a\}$, and $R^A = \{(a, a)\}$. Now the first claim holds, as all $x$ are in $P$. The second one holds as $R \neq \emptyset$. Finally the last one holds as both $P(a)$ and $Q(a)$ hold. Yet $P^A \neq Q^A$.

5. (a) (1 point) Give an example of a number that is in $\mathbb{R}$, but not in $\mathbb{Q}$.

> **Answer:** For example $\pi$, but any irrational will do.

(b) (4 points) Consider the following description of the predicate $P$ which should describe if a number is prime.
$\forall x \in \mathbb{N}(P(x) \leftrightarrow \forall n \in \mathbb{N}(n > 4 \rightarrow n \nmid x))$
This description is incorrect. Give one example of a number that has the property $P$ even though it is not prime, and give one example of a number that does not have the property $P$ even though it is prime. Explain both of your answers in at most 5 lines each.

> **Answer:** For instance $P(4)$ holds, even though $4$ is not prime, as all factors of $4$ are $1$, $2$, and $4$ which are all $\leq 4$. $P(1)$ also works. For instance $P(5)$ does not hold, as $5 \mid 5$ and $5 > 4$. Any prime $\geq 5$ will do here.

(c) (2 points) Consider the following claim: For all fractions $q = \dfrac{a}{b}$, if $a$ is even or if $b$ is even, then $P(q)$ and $R(q)$ hold. Consider now the following proof:

*Proof.*

1. Take arbitrary $q \in \mathbb{Q}$, such that $q = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $a$ or $b$ is even.
2. We now use a proof by division into cases.
3. Consider a case where $a$ is even.
4. <Derive that $P(q)$ holds.>
5. Consider a case where $b$ is even.
6. <Derive that $R(q)$ holds.>
7. Consider a case where both $a$ and $b$ are even.
8. <Derive that $P(q)$ and $R(q)$ hold.>
9. Since $q$ was arbitrary it holds for all $q = \dfrac{a}{b} \in \mathbb{Q}$ that if $a$ or $b$ is even, then $P(q)$ and $R(q)$ hold.

QED

This proof is not valid for the given claim. Explain why it is not valid and what we should change for the proof to become valid. Include references to the line numbers of the proof that should be changed. Answer in at most 8 lines.

> **Answer:** The proof is invalid, as we need to derive $P(q) \wedge R(q)$ in all cases. This means we need to derive more in both line 4 and line 6.
> Alternatively the proof as presented on the exam did not remark that $b \neq 0$ on line 1. If this is pointed out, this can also get full points.

6. (a) (8 points) Given a right angled triangle with sides $a$ and $b$ and hypotenuse[2] $c$, we know that Pythagoras' Theorem tells us that $a^2+b^2=c^2$. Use a proof by contradiction to prove that for all right-angled triangles $a+b>c$.

**Answer:**

*Proof.* Assume there is a right-angled triangle, s.t. $a+b \leq c$. Since all lengths are positive, squaring both sides we get: $(a+b)^2 = a^2+2ab+b^2 \leq c^2$. We also know that Pythagoras tells us that $a^2+b^2=c^2$. So we can substitute that in, to get: $2ab+c^2 \leq c^2$. But since $a, b > 0$ (they are lengths of a side of a triangle) so this statement cannot be true. This means either Pythagoras' theorem was wrong, or our assumption was wrong. Since Pythagoras' theorem has been proven, it must have been our assumption. QED

(b) (5 points) Prove the following claim for all integers $n$: if $n$ is a prime larger than $2$, then $n+1$ is not prime.

**Answer:**

*Proof.* Take an arbitrary $p$ that is a prime larger than $2$, to prove $p+1$ is not prime.
We know that since $p$ is a prime, $2$ cannot be a divisor of $p$. Therefore $p$ is odd and $p=2c+1$ for some constant $c$. This means that $p+1=2c+2=2(c+1)=2k$ with $k=c+1$. Therefore $p+1$ is even. This means that it cannot be prime, as $p+1>2$ and $2 \mid p+1$.
Since $p$ was arbitrarily chosen, it holds for all primes $p>2$ that $p+1$ is not prime. QED

(c) (2 points) Give a counterexample to the following claim about all integers $n$: $3 \nmid n \rightarrow 4 \mid (n^3 + 4n^2+3n)$. Explain how your example disproves the claim in at most 5 lines.

**Answer:** Take for instance $n=2$. $2^3+4 \cdot 2^2 + 3 \cdot 2 = 8+16+6 = 30$. It holds that $3 \nmid 2$, but $4 \nmid 30$. This disproves the claim.
Examples of numbers that work: $-2, 2, 7, 10$.

7. (a) (2 points) Consider the following recursively defined formula:

$$f(n) = \begin{cases} \pi & \text{if } n \leq 0 \\ f(n-3)+\pi & \text{if } n \text{ is odd} \\ f(n/2)-\pi & \text{if } n \text{ is even} \end{cases}$$

Compute $f(0), f(2), f(9)$.

**Answer:** $\pi, \pi, 2\pi$

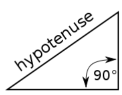
(b) Give a recursively defined sequence or formula for each of the following descriptions:

i. (3 points) All negative integers that end in a $1$.

**Answer:** $f_0=-1, f_n=f_{n-1}-10$ for $n>0$.

ii. (3 points) The first item of the sequence is the integer $2$, the second item is the first plus one. Every next item in the sequence is the sum of the previous two entries, plus the first item in the sequence.

**Answer:** $f_0=2, f_1=3, f_n=f_{n-1}+f_{n-2}+f_0$ for $n \geq 2$.

[2] Remember that the hypotenuse is the side opposite the right angle. Visually (image from Wikipedia): 


End of the exam