Midterm Reasoning and Logic (CSE1300)

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Please read the following information carefully!

- This exam consists of 8 open questions. The open questions are worth a total of 68 points and the points per open (sub) question are given in the (sub) question itself.
- The grade for this exam is computed as $1 + 9 \cdot \frac{\text{score}}{68}$. Note that you require a grade of at least 5 out of 10 to pass the course (assuming your average course grade is at least 5.75).
- This exam corresponds to chapters 1 to 3 of the book *Delftse Foundations of Computation* (version 1.1), with the exception of the topic of structural induction.
- You have 120 minutes to complete this exam.
- Before you hand in your answers, check that every sheet of paper contains your name and student number.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Read every question properly and in the case of the open questions, give **all information** requested. Do not however give irrelevant information, this could lead to a deduction of points.
- You may write on this exam paper and take it home.
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Question:	1	2	3	4	5	6	7	8	Total:
Points:	12	7	8	6	7	6	17	5	68

Learning goals coverage, based on the objectives of all lectures (strongly paraphrased):

Goal	mt 17	et 17	mc 18	mt 18	et 18	ret 18	mc 19	mt19
translate logic to and from natural language			1,2	3,4	1	1	1-2,19-20	3,4
describe $\wedge$,$\vee$,$\neg$,$\rightarrow$, and $\leftrightarrow$ operators	1						3	
construct a truth table	2,11a		3-5	1a,1b	31a	21a	4-5	1a
determine prop. logic equivalence	11b		6,7,19		2		6-8	1b
rewrite logical connectives	6		8-10		31b	21c	9	1c
describe contrapos, conv, and inv.	3		11,12			2	10	
describe logic validity			13,14		3		11,12	
describe sufficient and necessary conditions	4		15		4		13	2a
prove validty of argument in prop. logic	5		16-17	1b		3,21b	14	
describe the principle of explosion			18	1c			15	
explain why prop. logic is not suf. exp.			20					2b
describe $\forall$ and $\exists$ quantifiers	7,8		21	2c	5		19	
evaluate negation stmt. in pred. logic			22			4	17-18	
construct a Tarski's world			23-25				21-22	5b
construct a formal structure in pred. logic	12		26-27	2b	32a	22a	23	5a
evaluate claims about formal structures	12		28-29	2a	6,32b	22b	24	
construct counterexamples for claims	10		30	2a,2b,5c		5	25	6a
prove a predicate is satisfiable.								6b
describe the number sets $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$						6		
describe a proof by div. into cases				5b	7			
describe a proof by contradiction						7		7b
construct a proof by division into cases				7a				7b
construct a proof by contradiction	9							
describe a proof by contrapositive				5a				
construct a proof by contrapositive	13			7b				7a
describe a proof by generalisation.				5a,5b				
construct a proof by generalisation								
construct an existence proof					9			
identify proof to use for a given claim				5b				
compute a sequence of a rec. def.				6a	10	8		8
construct/interpret rec. def.		3		6b,6c				
explain the principle of an induction proof		2			11	9		
construct an ind proof for numbers					33a	23a		2c
construct an ind proof for algorithms		4			33b	10		
construct recursive definitions on sets		12a			12,13	23b,24		
construct a proof using struct. induction		12b			14,15	23c		
explain and apply basic set operations.		1			16	11		
construct Venn diagrams		5			17,18	12		
construct ce for claims on sets		1,13			19,34b	25		
compute the powerset of a set					20,21	13		
compute the cartesian product of two sets					22	14		
construct proofs for claims on sets					34a	25		
describe Cantor's proofs about infinite sets		11b			23	15		
construct f or R from nat. language					24,25	26a		
describe the diff. between f and R					35a	16		
determine the inverse of R and f		8			35b	26b		
determine if f is well-defined		6			26	17		
determine if f is inj., surj., or bij.		7,11a,c			27	18		
determine if R is sym., trans. or refl.		9			28,29	19		
describe an equivalence relation		10			30	20,26c		

1. (a) (2 points) Create a truth table for: $\neg(p \leftrightarrow q) \vee (\neg p \wedge q)$.

Answer:

p	q	$\overbrace{p \leftrightarrow q}^{A}$	$\neg A$	$\overbrace{(\neg p \wedge q)}^{B}$	$\neg A \vee B$
0	0	1	0	0	0
0	1	0	1	1	1
1	0	0	1	0	1
1	1	1	0	0	0

(b) Consider a new type of operator that operates on three propositions at a time. This ternary operator is defined by the following truth table:

p	q	r	$p \overset{q}{\rightarrow} r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Someone argues that $(q \overset{p}{\rightarrow} q) \vee r \equiv \neg(p \vee r) \overset{q}{\rightarrow} r$ holds.

i. (6 points) Create the full truth table for both propositions.

ii. (1 point) Describe how we can derive the (in)validity of this equivalence from your truth table.

Answer:

p	q	r	$\overbrace{q \overset{p}{\rightarrow} q}^{A}$	$A \vee r$	$\overbrace{\neg(p \vee r)}^{B}$	$B \overset{q}{\rightarrow} r$
0	0	0	0	0	1	1
0	0	1	0	1	0	1
0	1	0	0	0	1	0
0	1	1	0	1	0	1
1	0	0	0	0	0	0
1	0	1	0	1	0	1
1	1	0	1	1	0	0
1	1	1	1	1	0	1

As row one ($p = q = r = 0$) show, these propositions are not equivalent (left-hand side is false, right-hand side is true).

(c) (3 points) Rewrite $(p \vee q) \rightarrow \neg(r \wedge q)$ to DNF. Simplify your result as much as possible.

Answer:

$$\begin{aligned}(p \vee q) \rightarrow \neg(r \wedge q) &\equiv \neg(p \vee q) \vee \neg(r \wedge q) \\ &\equiv (\neg p \wedge \neg q) \vee \neg r \vee \neg q \\ &\equiv \neg r \vee \neg q\end{aligned}$$

Alternative through K-map:

p \ qr	00	01	11	10
0	1	1	0	1
1	1	1	0	1

2. For each of the following claims, either explain why they are true, or give a counterexample. Start your

answer with either the word "True" or "False" indicating which of the two options applies.

(a) (2 points) It is *not* possible for some proposition p to be both sufficient and necessary for some proposition q.

> **Answer:** False. This is perfectly possible, it is the bi-implication. $p \leftrightarrow q$ expresses exactly this.

(b) (2 points) Every argument that can be represented in propositional logic, can be represented in predicate logic.

> **Answer:** True. Take any proposition p, we make a predicate $P(x)$ that holds for some constant a iff p is true. This way we can we could for instance translate the argument: $p, p \to q \therefore q$ to $P(a), P(a) \to Q(b) \therefore Q(b)$.

(c) (3 points) You can prove a property $P(x)$ holds for all even integers ≥ 3 using induction. If you answer true, explain what the base case and induction step should look like. If you answer false, give a counterexample and a brief explanation.

> **Answer:** True. Take $n = 4$ for the base case, IH is it holds for arbitrary $P(k)$, now prove it holds for $P(k + 2)$.

3. For this question you need to translate claims from natural language to predicate logic. Make sure to define all predicates you introduce and to include all of the information from the original statement in your translation.

   (a) (2 points) Stefan loves CSE1300.

   Answer: $Loves(s, c)$
   s is Stefan, c is CSE1300, $Loves(x, y)$ is x loves y.

   (b) (3 points) There is a TA that has a blue badge.

   Answer: $\exists x(TA(x) \land \exists y(Blue(y) \land Badge(y) \land Has(x, y)))$
   Where all the predicates are what their name implies.

   (c) (3 points) When you grade an exam, you eat a pepernoot.[1]

   Answer: $\forall x((Exam(x) \land Grade(\text{you}, x)) \rightarrow \exists y(Pepernoot(y) \land Eat(\text{you}, y)))$
   Where you is you.

[1] "A pepernoot" is a small "cookie" Dutch people commonly eat from September to December.

4. Translate the following two claims to natural language (English). The predicates and domain used in our claims are as follows:

   - The domain of discourse is "all objects" (including living beings).
   - $Song(x)$ means: x is a song.
   - $Artist(x)$ means: x is an artist.
   - $Better(x, y)$ means: x is better than y.
   - $Performs(x, y)$ means: x performs y.
   - t is Tom Lehrer.
   - e is The Element Song.

   (a) (1 point) $Song(e) \land \neg Artist(e)$

   Answer: The element song is a song, not an artist.
   Grading rubric:

   - 1pt if both properties of the element song are correctly translated.

   (b) (1 point) $\neg \exists x(Artist(x) \land Better(x, t))$.

   Answer: There is no better artist than Tom Lehrer.
   Note that it does not mention Lehrer is an artist, thus technically speaking Tom Lehrer is the best artist is incorrect.

   (c) (4 points)
   $\forall x((Song(x) \rightarrow \exists y(Artist(y) \land Performs(y, x))) \land (Artist(x) \rightarrow \exists y(Song(y) \land Performs(x, y))))$

   Answer: All songs are performed by an artist and all artists perform a song.

5. (a) (4 points) Consider the following set of predicates over the domain $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

- $\forall x(P(x) \leftrightarrow (x \mid 2))$
- $\forall x(Q(x) \leftrightarrow \exists z(z < 4 \land (3 \mid (x + z))))$
- $\forall x \forall y(R(x, y) \leftrightarrow ((x = y + 2) \land \exists z((z > 4) \land (x \mid (z + y)))))$

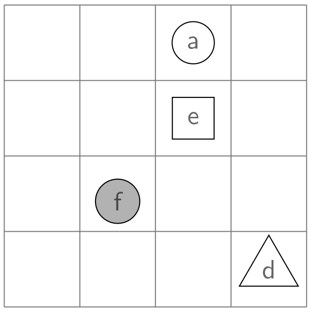
Give the truth sets for P, Q, and R.

Answer: $P = \{1, 2\}$, $Q = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and $R = \{(3, 1), (4, 2), (5, 3), (6, 4), (7, 5), (8, 6)\}$.

(b) (3 points) Draw a Tarski World with at least 4 objects and at most 6 objects, which satisfies the following criteria. Note that we have only two "colours": Filled (use whatever colour you are doing the exam with) and Empty (do not colour it). Briefly explain how your drawing satisfies the criteria.

- $\exists x(Square(x) \land \exists y(LeftOf(x, y)))$
- $Filled(f)$
- $\forall z(Filled(z) \to (\exists x(LeftOf(x, z) \land Triangle(x)) \lor \exists y(RightOf(y, z) \land Circle(z))))$

Answer:



6. (a) (3 points) Is the following argument valid? If so, explain why in at most 5 lines. If not, give a formal structure to prove it is not valid and briefly explain how your structure shows this.

$\forall x(P(x) \vee Q(x))$
$\forall x(P(x) \to \exists y(R(x,y)))$
$R(a,b)$
$\therefore P(a)$

Answer: This argument is false, as demonstrated by the formal structure S with $D^S = \{a, b\}$, $P^S = \{\}$, $Q^S = \{a, b\}$, $R^S = \{(a, b)\}$.

(b) (3 points) Is the following set of predicates satisfiable? If so, give a formal structure to prove it and briefly explain how your structure shows this. If not, explain why not in at most 5 lines. (Note that this is not the same set as in the previous question!)

$\forall x(P(x) \wedge Q(x))$
$\forall x(P(x) \to \forall y(R(x,y)))$
$\neg R(a,b)$

Answer: This is not satisfiable. The first statement says that $P(x)$ is true for all, so also for a. Thus $R(a,y)$ must hold for all y by the second premise. This contradicts the third one.

7. (a) (7 points) Prove the following claim for all integers n: if $n^2 + 6n - 3$ is even, then n is odd.

Answer:

Proof. Proof by contrapositive. We instread prove the logically equivalent statement: if n is even, then $n^2 + 6n - 3$ is odd.
Take an arbitrary k such that k is even. In other words $k = 2c$ for some integer c.
To prove: $k^2 + 6k - 3$ is odd.
$(2c)^2 + 6 \cdot (2c) - 3 = 4c^2 + 12c - 3 = 2(2c^2 + 6c - 2) + 1 = 2m + 1$ with $m = 2c^2 + 6c - 2$.
Thus $k^2 + 6k - 3$ is odd. Since k was arbitrarily chosen, it holds for all n that if $n^2 + 6n - 3$ is even, then n is odd. QED

(b) (7 points) Prove the following claim for all positive integers n: if $3 \nmid n$ then $3 \mid n^2 + 2$.

Answer:

Proof. Take an arbitrary k that is an integer such that $3 \nmid k$. That means we can split this proof into two cases:

- $k = 3c+1$. $k^2+2 = (3c+1)^2+2 = 9c^2+6c+1+2 = 9c^2+6c+3 = 3(3c^2+2c+1) = 2m$ for $m = 3c^2 + 2c + 1$. So $3 \mid k^2 + 2$ in this case.
- $k = 3c+2$. $k^2+2 = (3c+2)^2+2 = 9c^2+12c+4+2 = 9c^2+12c+6 = 3(3c^2+4c+2) = 2m$ for $m = 3c^2 + 2c + 1$. So $3 \mid k^2 + 2$ in this case.

Since k was arbitrarily chosen, it holds for all n that $3 \nmid n \rightarrow 3 \mid n^2 + 2$. QED

(c) (3 points) Consider the following faulty proof by contradiction for the claim: if n is prime, then $3n$ is prime.

Proof. We introduce $P(n)$ as the predicate for n is prime.

1. For the sake of contradiction, assume there is an integer k such that $\neg P(k) \land P(3k)$.
2. Since $\neg P(k)$ we know that there is some number c such that $c \mid k$.
3. Thus $k = c \cdot a$ for some integer a.
4. And $3k = 3c \cdot a = d \cdot a$, thus $d \mid 3k$.
5. Thus $\neg P(3k)$.
6. This forms a contradiction with our assumption in step 1, thus there cannot be such an integer k.
7. Thus $P(n) \to P(3n)$ holds for all n.

QED

Which of the steps contain(s) a mistake? Clearly describe what the mistake(s) is (are).

Answer: The claim is clearly false. $3c$ can never be prime! The mistake in the proof happens in step 1 in assuming the negation of the claim. The claim is: $\forall x(P(x) \to P(3x))$, the negation of this claim is: $\neg\forall x(P(x) \to P(3x)) \equiv \exists x(P(x) \land \neg P(3x))$. Thus the negation symbol was in the wrong place!

Some students pointed out that the second step should also contain that $c \neq 1$ and $c \neq k$, which is correct, but is not such a fundamental flaw in the proof as the one made in step 1. Partial points can be obtained for pointing out this flaw however.

8. (a) (2 points) Consider the following recursively defined formula:

$$f(n) = \begin{cases} 2^n & \text{if } n \leq 0 \\ 2 * f(n-1) & \text{if } n \text{ is odd} \\ f(n-2) + f(n/2) & \text{if } n \text{ is even} \end{cases}$$

Compute $f(-1), f(4), f(11)$.

Answer: $f(-1) = 0.5$
$f(4) = f(2) + f(2) = 2(f(0) + f(1)) = 2(3f(0)) = 6$
$f(11) = 2(f(10)) = 2(f(8) + f(5)) = 2(f(6) + 3f(4)) = 2(f(4) + f(3) + 3f(4)) = 2(4f(4) + 2f(2)) = 2(5(f(4))) = 10 \cdot 6 = 60$

(b) (3 points) Consider the following two formula:

$$f(n) = \sum_{i=1}^{n}(i+n) \qquad g(n) = \sum_{i=1}^{n} f(i)$$

Compute $f(4)$ and $g(3)$.

Answer: $f(4) = \sum_{i=1}^{4}(i+4) = 5+6+7+8 = 26$
$g(3) = \sum_{i=1}^{3} f(i) = f(1) + f(2) + f(3) = (1+1) + (1+2+2+2) + (1+3+2+3+3+3) = 2 + 7 + 15 = 24$

End of the exam