# Midterm Reasoning and Logic (CSE1300)

2021–22

**Please read the following information carefully!**

- This exam consists of 11 open questions. The open questions are worth a total of 54 points and the points per open (sub) question are given in the (sub) question itself.
- The grade for this exam is computed as $1 + 9 \cdot \frac{\text{score}}{54}$. Note that you require a grade of at least 5 out of 10 to pass the course (assuming your average course grade is at least 5.75).
- This exam corresponds to chapters 1 to 3 of the book *Delftse Foundations of Computation* (version 2.0), with the exception of the topic of structural induction.
- You have 120 minutes to complete this exam.
- Before you hand in your answers, check that every sheet of answer paper contains your name and student number.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Read every question carefully and, in the case of the open questions, give **all information** requested. Do not however give irrelevant information: this could lead to a deduction of points.
- You may write on this exam paper and take it home.
- No marmots and/or hippos were harmed in the creation of this exam.
- Exam prepared by S. Hugtenburg, S. Dumančić, I. van Kreveld, N. Yorke-Smith
- Exam is ©2021 TU Delft.

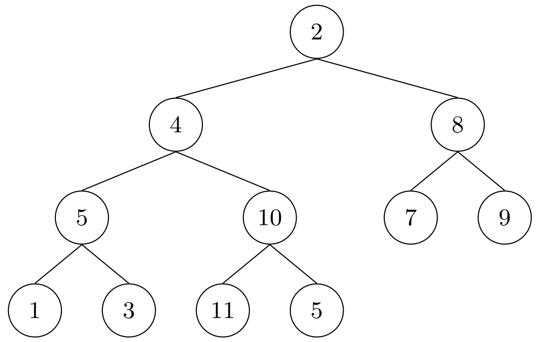
| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | Total: |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 3 | 10 | 4 | 2 | 5 | 2 | 5 | 10 | 3 | 2 | 8 | 54 |

Learning goals coverage, based on the objectives of all lectures (strongly paraphrased):

| Goal | mc 18 | mt 18 | et 18 | rt 18 | mc 19 | mt 21 | et 21 | rt 21 |
|---|---|---|---|---|---|---|---|---|
| translate logic to and from natural language | 1,2 | 3,4 | 1 | 1 | 1,16-17 | 3-4 | | |
| describe $\wedge$,$\vee$,$\neg$,$\rightarrow$, and $\leftrightarrow$ operators | | | | | 2 | | | |
| construct a truth table | 3-5 | 1a,1b | 31a | 21a | 3-4 | 1a | | |
| determine prop. logic equivalence | 6,7,19 | | 2 | | 5-7 | 1b | | |
| rewrite logical connectives | 8-10 | | 31b | 21c | 8 | 1c,d | | |
| describe contrapositives, converses, and inverses. | 11,12 | | | 2 | 9 | | | |
| describe logic validity | 13,14 | | 3 | | 10,11 | | | |
| describe sufficient and necessary conditions | 15 | | 4 | | 12 | 1c | | |
| prove validty of argument in prop. logic | 16-17 | 1b | | 3, 21b | | | | |
| describe the principle of explosion | 18 | 1c | | | 13 | | | |
| explain why prop. logic is not suf. expressive | 20 | | | | | | | |
| describe $\forall$ and $\exists$ quantifiers | 21 | 2c | 5 | | | | | |
| evaluate negation stmt. in pred. logic | 22 | | | 4 | 14-15 | | | |
| construct a Tarski's world | 23-25 | | | | | | | |
| construct a formal structure in pred. logic | 26-27 | 2b | 32a | 22a | 18 | 2a | | |
| evaluate claims about formal structures | 28-29 | 2a | 6,32b | 22b | 19 | 2b | | |
| construct counterexamples for claims | 30 | 2a,2b,5c | | 5 | 20 | 2c | | |
| describe the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ | | | | 6 | | | | |
| describe the form of a proof by div. into cases | | 5b | 7 | | | 5a | | |
| describe the form of a proof by contradiction | | | | 7 | | | | |
| construct a proof by division into cases | | 7a | | | | | | |
| construct a proof by contradiction | | | | | | 5b | | |
| explain what a theorem prover is. | | | 8 | | | | | |
| describe the form of a proof by contrapositive | | 5a | | | | | | |
| construct a proof by contrapositive | | 7b | | | | | | |
| describe the form of a proof by generalisation. | | 5a,5b | | | | | | |
| construct a proof by generalisation | | | | | | | | |
| construct an existence proof | | | 9 | | | | | |
| identify type of proof to use for a given claim | | 5b | | | | 5b,8 | | |
| compute a sequence given a recursive definition | | 6a | 10 | 8 | | 7a | | |
| construct and interpret recursive definitions | | 6b,6c | | | | | | |
| explain the basic principle of an induction proof | | | 11 | 9 | | 6a | | |
| construct an ind proof for numbers | | | 33a | 23a | | 6b | | |
| construct a tree based on a description | | | | | | 7b | | |
| construct an ind proof for algorithms | | | 33b | 10 | | | | |
| construct recursive definitions on sets | | | 12,13 | 23b, 24 | | | | |
| construct a proof using structural induction | | | 14,15 | 23c | | | | |
| explain and apply basic set operations. | | | 16 | 11 | | | | |
| construct Venn diagrams | | | 17,18 | 12 | | | | |
| construct counterexamples for claims on sets | | | 19,34b | 25 | | | | |
| compute the powerset of a set | | | 20,21 | 13 | | | | |
| compute the cartesian product of two sets | | | 22 | 14 | | | | |
| construct proofs for claims on sets | | | 34a | 25 | | | | |
| describe Cantor's proofs about infinite sets | | | 23 | 15 | | | | |
| construct f or R from nat. language | | | 24,25 | 26a | | | | |
| describe the diff. between f and R | | | 35a | 16 | | | | |
| determine the inverse of R and f | | | 35b | 26b | | | | |
| determine if f is well-defined | | | 26 | 17 | | | | |
| determine if f is injective, surjective, or bijective | | | 27 | 18 | | | | |
| determine if R is symmetric, transitive or reflexive | | | 28,29 | 19 | | | | |
| describe the properties of an equivalence relation | | | 30 | 20, 26c | | | | |

1. (3 points) Give a binary tree with the following properties (a visualisation suffices):
   - The root has an even value.
   - The leaves have odd values.
   - The height of the tree is at most $5$.
   - The tree has 10 unique values.
   - In the in-order traversal of the tree, the sequence $7 - 8 - 9$ appears.

**Answer:**




**Grading rubric:**

- 1pt if at least root and leaves properties satisfied.
- 1pt if at least in-order traversal property satisfied.
- 1pt if all other properties are satisfied

2. (a) (4 points) Construct a truth table for the following two propositional formulae.
   i. $\neg r \rightarrow (p \vee q)$
   ii. $p \leftrightarrow \neg(q \wedge p)$

**Answer:**

| $p$ | $q$ | $r$ | $\neg r$ | $\rightarrow$ | $(p$ | $\vee$ | $q)$ | $p$ | $\leftrightarrow$ | $\neg($ | $q$ | $\wedge$ | $p)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |

**Grading rubric:**

- -1pt per incorrect operator: $\neg, \rightarrow, \vee, \leftrightarrow, \wedge$ (at most -2pt per statement).
- If only final answer is given for a statement, either 0 or 2 points.

(b) (1 point) Consider again the two propositional formulae from the previous subquestion. Are they equivalent? Explain your answer by referring to (specific rows from) your truth table.

**Answer:** They are not, for example in the second row the values are different (when $p = q = 0, r = 1$).
**Grading rubric:**

- 1pt Correct counterexample by mentioning row or values.

(c) (2 points) Consider a new ternary operator $p \xrightarrow{q} r$, such that when $q$ is false $p \xrightarrow{q} r$ is equivalent to $p \leftrightarrow r$, and when $q$ is true $p \xrightarrow{q} r$ is equivalent to $p \wedge \neg r$. Give a truth table for $p \xrightarrow{q} r$.

**Answer:**

| $p$ | $q$ | $r$ | $p \xrightarrow{q} r$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

**Grading rubric:**

- 1pt for $q = 0$ giving the right results.
- 1pt for $q = 1$ giving the right results.

(d) (3 points) The operator $p \xrightarrow{q} r$ is functionally complete. Prove this.

**Answer:** The shortest way to do this is to show that we can emulate another functionally complete set with just this operator. We show that we can emulate $\neg$ with this operator by just feeding it the same input three times.

| $p$ | $\neg p$ | $p \xrightarrow{p} p$ |
|---|---|---|
| 0 | 1 | 1 (row 1 above) |
| 1 | 0 | 0 (row 8 above) |

Now we observe that $p \xrightarrow{\neg p} \neg q$ has the same truth table as $p \wedge q$.

| $p$ | $q$ | $p \wedge q$ | $p \xrightarrow{\neg p} \neg q$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 (row 4 above) |
| 0 | 1 | 0 | 0 (row 3 above) |
| 1 | 0 | 0 | 0 (row 5 above) |
| 1 | 1 | 1 | 1 (row 6 above) |

**Grading rubric:**

- 1pt for correct strategy (taking a different set, or trying to create 16 truth tables...)
- 1pt for correctly showing $\neg$ can be done (or NAND or NOR in which case both this one and the next).
- 1pt for correctly creating $\wedge$ or $\vee$ (or NAND or NOR)

3. (4 points) Provide a formal structure with a domain of at least $3$ and at most $6$ elements that satisfies the following set of predicate statements:

- $P(a) \wedge Q(z)$
- $\forall x(P(x) \vee Q(x))$
- $\exists x \exists y(P(x) \wedge \neg Q(x) \wedge x \neq y \wedge P(y) \wedge \neg Q(y))$
- $\forall x \exists y(R(x, y) \wedge \neg(P(y) \wedge Q(y)))$

**Answer:** Consider the following formal structure $S$ with $D = \{a, b, z\}$

- $P^S = \{a, b\}$
- $Q^S = \{z\}$
- $R^S = \{(a, b), (b, b), (z, b)\}$

**Grading rubric:**

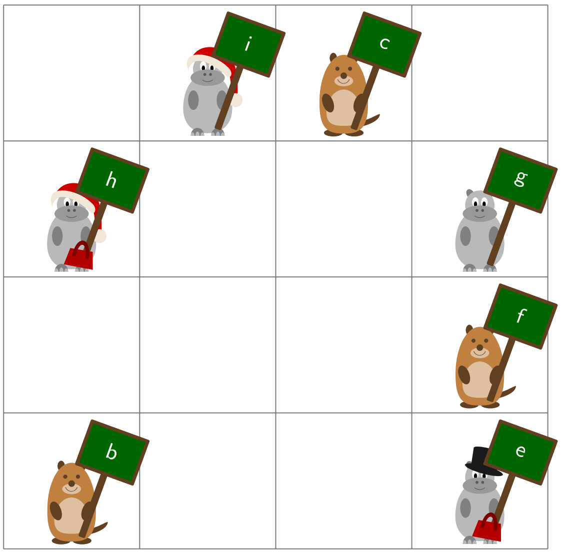
- 1pt per statement made true.
- -1pt if $R$ is not a set of tuples.
- -1pt if no domain is given (this matters for the second and fourth, but we're already penalising the 4th with the set of tuples).

4. (2 points) Consider the following animal world filled with marmots and hippos. The signs the animals are holding represent the object they are in the world. For example the hippo in the bottom right corner is $e$ and the marmot in the bottom left corner is $b$.

   Due to a new law imposed on the animal world, it is now a punishable offense to break the following law. In this law the predicates $AboveOf$ and $LeftOf$ work just as in a Tarski World (so $AboveOf(x, y)$ means $x$ is above of $y$). The predicates $Tophat(x)$, $Santa(x)$, and $Handbag(x)$ mean that $x$ has a tophat, santa hat, or handbag respectively. Finally $Marmot(x)$ means that $x$ is a marmot, and $Hippo(x)$ means $x$ is a hippo.

   By the order of the emperor, the following law is in effect immediately:
   $\forall x((Marmot(x) \land \exists y(AboveOf(x, y) \land Tophat(y))) \leftrightarrow \exists y(Handbag(y) \land LeftOf(x, y)))$

   Which creature(s) should get a fine, and why?




**Answer:** $f$ should get a fine as it is a marmot above a creature with a tophat (e), but there is no creature with a handbag so that $f$ is left of that creature. Similarly for $b$, $h$, and $i$ there is a creature with a handbag so that they are to the left of it, but since they are not marmots above a tophat or even marmots at all (the "and only if" part!), they should also be fined! **Grading rubric:**

- 1pt for $f$, including explanation.
- 1pt for $b$ and $h$, $i$, including explanation.
- alternatively: 1pt for all correct, 1pt for correct explanation
- -1pt for any animal fined that shouldn't be fined.

5. Translate the following two claims to natural language (English). The predicates and domain used in our claims are as follows:

- The domain of discourse is ‘all objects’ (including living beings).
- $Fork(x)$ means: $x$ is a fork.
- $Spoon(x)$ means: $x$ is a spoon.
- $Soup(x)$ means: $x$ is soup.
- $Eats(x, y)$ means: $x$ eats $y$.
- $Eats(x, y, z)$ means: $x$ eats $y$ using $z$.
- $s$ is Stradivari.
- $n$ is Nigel.

(a) (1 point) $\exists x(Fork(x) \land Spoon(x))$

**Answer:** There is a fork that is also a spoon. **Grading rubric:**

- 1pt correct translation

(b) (2 points) $\neg\exists x((Fork(x) \lor Spoon(x)) \land \exists y(Eats(s, y, x)) \land \exists y(Eats(n, y, x)))$

**Answer:** There is no fork or spoon that both Stradivari and Nigel eat something with. **Grading rubric:**

- 1pt There is no fork or spoon.
- 1pt that Stradivari and Nigel both eat with.

(c) (2 points) $\forall x((Soup(x) \land Eats(s,x)) \to \exists a(Fork(a) \land Eats(s,x,a)))$

**Answer:** Stradivari eats all of his soup using a fork. **Grading rubric:**

- 1pt correct if/then present in the translation, that keeps $s$ throughout.
- 1pt correctly realise that it's the same x.
- -1pt if the English translation implies soup exists (ambiguity about empty domain in Zesje)

6. (2 points) Compute $f(4)$ and $f(7)$, where $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ f(\frac{x-3}{2}) + 3 & \text{if } 2 \nmid x \\ 2f(2x+1) & \text{else} \end{cases}$

Show your computation.

**Answer:** $f(4) = 2f(9) = 2(f(3) + 3) = 2(f(0) + 3 + 3) = 2(9) = 18$
$f(7) = f(2) + 3 = 6$
**Grading rubric:**

- 1pt for each
- -1pt if no computations are shown.

7. Translate the following English phrases to predicate logic. Make sure to define all predicates and constants you use in your translation. You may only use the existential and universal quantifiers, the logical connectives, and any predicates and constants you define.

(a) (1 point) Amy is a hippo.

**Answer:** $Hippo(a)$ where $a$ is Amy and $Hippo(x)$ means $x$ is a hippo. **Grading rubric:**

- 1pt if correct.

(b) (2 points) There is a marmot that likes Amy.

**Answer:** $\exists x(Marmot(x) \land Likes(x,a))$ where $Marmot(x)$ is for $x$ is a marmot and $Likes(x,y)$ is for $x$ likes $y$. **Grading rubric:**

- 1pt for quantifier
- 1pt for correct connective and predicates.

(c) (2 points) Not all marmots that like Amy are also liked by Amy.

**Answer:** $\neg\forall x((Marmot(x) \land Likes(x,a)) \to Likes(a,x))$
Or equivalently: $\exists x(Marmot(x) \land Likes(x,a) \land \neg Likes(a,x))$ **Grading rubric:**

- 1pt: Correct conjunction of marmot and liking amy.
- 1pt: Correct forall with an implication or there exists with conjunction.

8. (a) (3 points) Someone wants to prove the following claim using induction:
For all integers $n$ that are a power of 2: $P(n)$ holds.
Describe what the base case and induction hypothesis should look like, and what should be proven in the inductive step.

**Answer:** In the base case we should prove $P(1)$ (which is $2^0$). The IH is that $P(k)$ holds for some integer $k$ that is a power of two. Now we need to prove it holds for $P(2k)$. **Grading rubric:**

- 1pt correct base case ($P(2)$ is also considered correct, though a neutral remark should be given)

- 1pt correct IH.
- 1pt correct inductive step.

(b) (7 points) Consider a variation on the Fibonacci sequence, called the Tribonacci Sequence, with: $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 4$.
Prove using mathematical induction that for all $n \geq 1$: $T_n \leq 2^n$.

**Answer:**

*Proof.* Base case ($n = 1$): $T_1 = 1 \leq 2 = 2^1$
Base case ($n = 2$): $T_2 = 1 \leq 4 = 2^2$
Base case ($n = 3$): $T_3 = 1 \leq 8 = 2^3$

Take an arbitrary $k \geq 3$ such that for all $j \leq k$ it holds that $T_j < 2^j$. To prove $T_{k+1} < 2^{k+1}$.

$$\begin{aligned} T_{k+1} &= T_k + T_{k-1} + T_{k-2} \\ (IH) &< 2^k + 2^{k-1} + 2^{k-2} \\ &= 4 \cdot 2^{k-2} + 2 \cdot 2^{k-2} + 2^{k-2} \\ &= 7 \cdot 2^{k-2} \\ &< 8 \cdot 2^{k-2} \\ &= 2^{k+1} \end{aligned}$$

Since $k$ was arbitrarily chosen it holds for all $k \geq 3$. Thus by the principle of induction it holds for all $n$ that $T_n < 2^n$. QED

**Grading rubric:**

- 1pt at least one base case correct
- 1pt all base cases correct
- 1pt IH correct for weak induction (including the notion of arbitrary)
- 1pt IH correct for strong induction (including the notion of arbitrary)
- 1pt Apply IH correctly
- 1pt The rest of the math
- 1pt Structuring the proof according to our template + wrapping up the proof

9. (3 points) Having previously recovered the four sacred elemental crystals, a new set of thieves has taken it upon themselves to steal them! Amy the hippo, Maximillion (Max for short) the marmot, Paya, and Luke have all eaten one of the crystals and now claim the following:

- Amy says that Paya ate the water crystal.
- Max says he doesn't remember anything.
- Paya says that she didn't eat the earth crystal
- Luke says that if Max ate the air crystal, then Paya ate the earth crystal

Remember that:

- Each person ate a different elemental crystal
- The earth and water crystals force their owner to tell the truth
- The fire and air crystals force their owners to lie

Who ate what crystal? Explain how you derived your answer.

**Answer:** Suppose Paya would be lying. Then, she ate the fire crystal or the air crystal, so she didn't eat the earth crystal. But then Paya is telling the truth, which is a contradiction. Therefore, Paya is telling the truth. Therefore, Paya did not eat the earth crystal, and also not the air or fire crystal, so Paya ate the water crystal.

As a result, Amy is telling the truth! Hence she must have eaten the earth crystal!

That just leaves our lying crystals. Max doesn't help in this regard, but in order to force Luke into a lie, the first part of the if-then must be true and the second part false. Thus Max ate the air crystal (and indeed Paya ate water, not earth!), and Luke ate the fire crystal.

**Grading rubric:**

- 1pt for correctly explaining why Paya must have eaten the water crystal
- 1pt for correctly explaining that Max and Luke must be lying.
- 1pt for a correct final answer (even without explanation as to how it was derived, e.g., when it is guessed and then explained.)

10. (2 points) Consider the following argument. If it is valid, start your answer with '*Valid*' and explain why it is true. If it is invalid, start your answer with '*Invalid*' and explain why it is not valid by providing a counterexample and an explanation showing that your counterexample is indeed a counterexample.

$$\begin{array}{l} \forall x(P(x) \land Q(x)) \\ \forall x(P(x) \rightarrow \exists y(R(y,x))) \\ \hline \therefore \exists x \exists y(R(x,y)) \end{array}$$

**Answer:** Take a structure with an empty domain. Now both premises are vacuously true. The conclusion however is false as $R$ must be empty since $D$ is empty. **Grading rubric:**

- 1pt take the empty domain.
- 1pt explain why this works.
- For different answers, 1pt if both premises true, 1 pt if conclusion is made false.

11. (a) (2 points) Consider the following proof by division into cases for the claim:

For all integers $n$: if 5 does not divide $n$ then $P(n)$ holds.

*Proof.* Take an arbitrary integer $k$ such that $5 \nmid k$. Now we exhaustively divide this into the following cases:

- $k = 5m + 1$ for some integer $m$, $\langle$ do math here $\rangle$ therefore $P(k)$ holds.
- $k = 5m + 3$ for some integer $m$, $\langle$ do math here $\rangle$ therefore $P(k)$ holds.
- $k = 2m$ for some integer $m$, $\langle$ do math here $\rangle$ therefore $P(k)$ holds.

Since it holds in all cases and $k$ was arbitrarily chosen it now holds for all $n$ that if $5 \nmid n$ then $P(n)$ holds. QED

This proof is flawed. Indicate exactly where the proof goes wrong and give a clear example of an integer $n$ for which this claim should hold but this proof does not show it.

**Answer:** The cases are not exhaustive. All odd numbers with a remainder of 2 or 4 when dividing by 5 are not included (e.g. 7 cannot be written as $5m + 1$, $5m + 3$ or $2m$ for any integer $m$). **Grading rubric:**

- 1pt Correctly point out why the division into cases is wrong.
- 1pt Correctly give a specific counterexample.

(b) (6 points) Prove the following claim:

For all real numbers $x$ and $y$ with $x \neq 0$ and $y \neq 0$, it holds that $\sqrt{x^2 + y^2} \neq x + y$.

**Answer:**

*Proof.* Consider a proof by contradiction. Take some real numbers $x \neq 0$ and $y \neq 0$ such that

$\sqrt{x^2+y^2} = x+y$. Now we get:

$$\begin{aligned}\sqrt{x^2+y^2} &= x+y\\ x^2+y^2 &= (x+y)^2\\ x^2+y^2 &= x^2+2xy+y^2\\ 0 &= 2xy\\ 0 &= xy\end{aligned}$$

Since $x \neq 0$ we can divide both sides by $x$ and we get $0 = y$ . This contradictions our assumption that $y \neq 0$. Hence no such $x, y$ can exist and thus for all real numbers $x, y$ it must hold that if $x \neq 0$ and $y \neq 0$ then $\sqrt{x^2+y^2} \neq x+y$. QED

**Grading rubric:**

- 2pt correct start of a proof by contradiction (assume first part true, second false), or correct start of a proof by contrapositive (correct contrapositive).
- 1pt take some real number (for contradiction) or arbitrary number (for contrapositive)
- 1pt Square both sides, work out the square
- 1pt conclude correctly that $x$ or $y$ must be equal to zero.
- 1pt Wrap up the proof correctly

End of the exam